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DEFENCE SCIENCE AND TECHNOLOGY ORGANISATION

ELECTRONICS RESEARCH LABORATORY

SALISBURY, SOUTH AUSTRALIA

SPECIAL DOCUMENT

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FIELD ANALYSIS AND POTENTIAL THEORY
PART 2

R.S. EDGAR

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PREFACE

Part 2 of "Field Analysis and Potential Theory" complements the treatment of Maxwell's equations for systems of doublets and whirls at rest, as presented in Part 1. Considerations are extended to time-dependent configurations of doublets and whirls in uniform translation, and the appropriately-modified forms of Maxwell's equations and the boundary conditions are developed. As in Part 1 the Liénard-Wiechert potentials are taken to be fundamental.

R.S. Edgar

NOTATION

As in Part 1,

(1) vector quantities are represented by a bar over the associated symbol.

(2) unit vectors, other than $\bar{i}, \bar{j}, \bar{k}$, carry a circumflex superscript.

(3) Gaussian units are employed throughout.

Equation numbering duplicates that of Chapter 1, Part 1, but no confusion arises since all cross references refer to other chapters.

CONTENTS

CHAPTER 1

THE EXTENSION OF MAXWELL'S EQUATIONS TO DOUBLET AND WHIRL CONFIGURATIONS IN UNIFORM MOTION

1.1 Introduction

The analyses of Part 1 have been restricted to considerations of doublet and whirl complexes at rest. In Part 2 we extend considerations to such configurations in uniform motion. The analysis will be carried out by making use of a purely algebraical mapping technique involving space and time transformations, which permits of an appropriate extension of Maxwell's equations. Like their earlier counterparts, these equations continue to be analytical consequences of certain definitions[1].

1.2 The Lorentz Transformation

Consider a configuration of point sources of which the typical source occupies the position P (x, y, z) at the time t. From this configuration we fabricate a hypothetical counterpart by matching the source at P by one of equal strength at P′ (x′, y′, z′) at the time t′, where

$$x' = \beta(x-vt)\ ;\ y' = y\ ;\ z' = z\ ;\ t' = \beta(t-vx/c^2) \qquad (1.2\text{-}1)$$

$$\beta = (1-v^2/c^2)^{-1/2}$$

v being an arbitrary constant of magnitude less than c.

It is seen that in this particular transformation (known as the Lorentz transformation) the space and time coordinates of a given source in the secondary configuration (denoted S′) are each dependent upon its space and time coordinates in the primary system (denoted S).

<u>The same units of space and time obtain in each configuration</u>.

The points P and P′, when associated respectively with the times t and t′, are said to be conjugate.

Suppose that the typical source moves from its position x, y, z at time t to the position $x+\Delta x$, $y+\Delta y$, $z+\Delta z$ at time $t+\Delta t$. Its components of velocity (u_x, u_y, u_z) are the limiting values of $\Delta x/\Delta t$, $\Delta y/\Delta t$, $\Delta z/\Delta t$ as $\Delta t \to 0$. In S′ the matching source moves from the position $\beta(x-vt)$, y, z at time $\beta(t-vx/c^2)$ to the position $\beta(x+\Delta x-v(t+\Delta t))$, $y+\Delta y$, $z+\Delta z$ at time $\beta(t+\Delta t-v(x+\Delta x)/c^2)$, whence

1. We are not concerned here with Einsteinian relativity. However, a nexus between the special theory and the present analysis is provided in the exercises at the end of the chapter.

$$u'_x = \lim_{\Delta t \to 0} \frac{\beta(\Delta x - v\Delta t)}{\beta(\Delta t - v\Delta x/c^2)} = \frac{u_x - v}{(1 - vu_x/c^2)} \qquad (1.2\text{-}2)$$

$$u'_y = \lim_{\Delta t \to 0} \frac{\Delta y}{\beta(\Delta t - v\Delta x/c^2)} = \frac{u_y}{\beta(1 - vu_x/c^2)} \qquad (1.2\text{-}3)$$

$$u'_z = \lim_{\Delta t \to 0} \frac{\Delta z}{\beta(\Delta t - v\Delta x/c^2)} = \frac{u_z}{\beta(1 - vu_x/c^2)} \qquad (1.2\text{-}4)$$

We may proceed in the same way to determine the relationships between the components of acceleration of matching sources in S and S'. We find that

$$\frac{du'_x}{dt'} = \frac{du_x}{dt} \frac{1}{\beta^3(1 - vu_x/c^2)^3} \qquad (1.2\text{-}5)$$

$$\frac{du'_y}{dt'} = \frac{du_y}{dt} \frac{1}{\beta^2(1 - vu_x/c^2)^2} + \frac{du_x}{dt} \frac{vu_y/c^2}{\beta^2(1 - vu_x/c^2)^3} \qquad (1.2\text{-}6)$$

$$\frac{du'_z}{dt'} = \frac{du_z}{dt} \frac{1}{\beta^2(1 - vu_x/c^2)^2} + \frac{du_x}{dt} \frac{vu_z/c^2}{\beta^2(1 - vu_x/c^2)^3} \qquad (1.2\text{-}7)$$

Since the components of velocity in S' are dependent only upon those in S and are independent of space and time coordinates, a system of sources which translates as a whole in S gives rise to a similar system in S' although, in general, the configurations will be of different shape.

When (1.2-1) is solved for x, y, z, t in terms of x', y', z', t' we obtain

$$x = \beta(x' + vt') \ ; \ y = y' \ ; \ z = z' \ ; \ t = \beta(t' + vx'/c^2) \qquad (1.2\text{-}8)$$

ie the reverse transformation is derived from the forward transformation by transposition of the primed and unprimed letters and reversal of the sign of v.

Combinations which transform in this way are said to comprise the components of a four-vector.

1.3 Transformation of the Microscopic Retarded Potentials

Suppose that within the S configuration a source of strength a, located at Q (x_1, y_1, z_1) at the time t_1, is in the appropriately retarded position for evaluation of its potentials at O (x_0, y_0, z_0) at the time t_0. Then $t_1 = t_0 - R/c$ where R is the distance of O from Q (Fig. 1.1), hence $R = c(t_0 - t_1)$

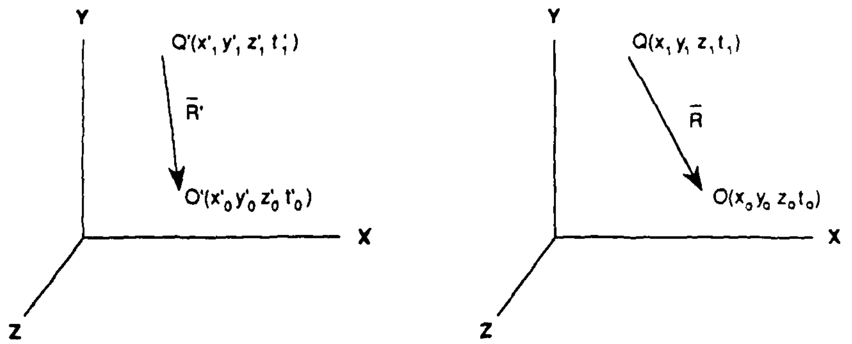

Fig. 1.1

If the matching source within S' is located at Q' (x_1', y_1', z_1') at the time t_1' and the conjugate point of evaluation at O' (x_0', y_0', z_0') at the time t_0', and if Q'O'= R', then

$$
\begin{aligned}
R'^2 &= (x_0'-x_1')^2 + (y_0'-y_1')^2 + (z_0'-z_1')^2 \\
&= \beta^2 \left\{x_0-x_1-v(t_0-t_1)\right\}^2 + (y_0-y_1)^2 + (z_0-z_1)^2 \\
&= R^2 + (\beta^2-1)(x_0-x_1)^2 + \beta^2 v^2 (t_0-t_1)^2 - 2\beta^2 v(x_0-x_1)(t_0-t_1) \\
&= (c^2+\beta^2 v^2)(t_0-t_1)^2 + \beta^2 \frac{v^2}{c^2}(x_0-x_1)^2 - 2\beta^2 v(x_0-x_1)(t_0-t_1) \\
&= c^2\beta^2 \left\{t_0-t_1-\frac{v}{c^2}(x_0-x_1)\right\}^2 \\
&= c^2(t_0'-t_1')^2
\end{aligned}
$$

or $R' = c(t_0'-t_1')$

It follows that Q' is the appropriately retarded position of the matching source in S' for evaluation of its potentials at the point conjugate to O <u>for the same retardation constant c</u>.

If l,m,n and l',m',n' are the direction cosines of $\bar{R}$ and $\bar{R}'$, then

$$R' = c(t_0'-t_1') = c\beta \left\{t_0-t_1-\frac{v}{c^2}(x_0-x_1)\right\} = c\beta \left\{\frac{R}{c} - \frac{v}{c^2} lR\right\}$$

$$\text{or} \quad R' = \beta R\left(1-\frac{lv}{c}\right) \tag{1.3-1}$$

Also $l' = \dfrac{x_0'-x_1'}{R'} = \beta\left\{\dfrac{x_0-x_1 - v(t_0-t_1)}{R'}\right\} = \dfrac{\beta\,(lR-vR/c)}{\beta\, R(1-lv/c)}$

$$\text{or} \quad l' = \frac{1-v/c}{1-lv/c} \tag{1.3-2}$$

Further, $m' = \dfrac{y_0'-y_1'}{R'} = \dfrac{y_0-y_1}{\beta\, R(1-lv/c)}$

$$\text{or} \quad m' = \frac{m}{\beta(1-lv/c)} \tag{1.3-3}$$

Similarly
$$n' = \frac{n}{\beta(1-lv/c)} \tag{1.3-4}$$

If ϕ' and $\bar{A}'$ are the retarded scalar and vector potentials at x_0', y_0', z_0', t_0' of the source at Q′, then

$$\phi' = \frac{a}{R'(1-u'_{R'}/c)} \quad \text{and} \quad \bar{A}' = \frac{a\bar{u}'}{cR'(1-u'_{R'}/c)} \tag{1.3-5}$$

Substitution for l', m', n' and u_x', u_y', u_z' in $u'_{R'} = l'u_x' + m'u_y' + n'u_z'$ leads to the relationship

$$(1-u'_{R'}/c) = (1-u_R/c)\Big/\beta^2\left(1-\frac{vu_x}{c^2}\right)\left(1-\frac{lv}{c}\right) \tag{1.3-6}$$

whence we find that

$$\phi' = \beta\phi\left(1-\frac{vu_x}{c^2}\right) = \beta\left(\phi - \frac{v}{c}A_x\right) \tag{1.3-7}$$

$$A_x' = \beta\left(A_x - \frac{v}{c}\phi\right) \;;\; A_y' = A_y \;;\; A_z' = A_z \tag{1.3-8}$$

where ϕ and $\bar{A}$ are the potentials at x_0, y_0, z_0, t_0 of the parent source in S.

It should be noted that these relationships hold for any type of source motion (ie accelerated or non-accelerated with $|u|<c$) and, by superposition, for any combination of point sources including doublets and whirls in motion.

The reverse transformations take the form

$$\phi = \beta\phi'\left(1+\frac{v}{c^2}u_x'\right) = \beta\left(\phi' + \frac{v}{c}A_x'\right) \tag{1.3-9}$$

$$A_x = \beta\left(A_x' + \frac{v}{c}\phi'\right) \;;\; A_y = A_y' \;;\; A_z = A_z' \tag{1.3-10}$$

It will be seen that $A_x, A_y, A_z, \phi/c$ comprise the components of a four-vector.

1.4 Transformation of the Microscopic $\bar{E}$ and $\bar{B}$ Fields.

Consider first the change of ϕ which accompanies a movement $\bar{i}\Delta x$ from 0 at the time t_0. This movement shifts the conjugate point away from 0′ by $\bar{i}\beta\Delta x$ in space and $-\beta v\Delta x/c^2$ in time, in accordance with (1.2.-1). The corresponding change of ϕ' is given by

$$\Delta\phi' \approx \frac{\partial\phi'}{\partial x'}\ \beta\Delta x - \frac{\partial\phi'}{\partial t'}\ \frac{\beta v}{c^2}\ \Delta x$$

where the derivatives are evaluated at 0′.

Similarly,

$$\Delta A'_x \approx \frac{\partial A'_x}{\partial x'}\ \beta\Delta x - \frac{\partial A'_x}{\partial t'}\ \frac{\beta v}{c^2}\ \Delta x$$

[These relationships may be written succinctly as

$$\frac{\partial}{\partial x} = \beta\left(\frac{\partial}{\partial x'} - \frac{v}{c^2}\ \frac{\partial}{\partial t'}\right) \tag{1.4-1}$$

for operation upon a primed quantity]

But since $\phi = \beta(\phi' + \frac{v}{c} A'_x)$ and $A_x = \beta(A'_x + \frac{v}{c}\phi')$ at conjugate points

$$\Delta\phi = \beta(\Delta\phi' + \frac{v}{c}\Delta A'_x) \quad \text{and} \quad \Delta A_x = \beta(\Delta A'_x + \frac{v}{c}\Delta\phi')$$

hence

$$\Delta\phi \approx \beta\left\{\frac{\partial\phi'}{\partial x'}\ \beta\Delta x - \frac{\partial\phi'}{\partial t'}\ \frac{\beta v}{c^2}\ \Delta x + \frac{v}{c}\left(\frac{\partial A'_x}{\partial x'}\ \beta\Delta x - \frac{\partial A'_x}{\partial t'}\ \frac{\beta v}{c^2}\ \Delta x\right)\right\}$$

and

$$\frac{\partial\phi}{\partial x} = \beta^2\left\{\frac{\partial\phi'}{\partial x'} - \frac{v}{c^2}\ \frac{\partial\phi'}{\partial t'} + \frac{v}{c}\ \frac{\partial A'_x}{\partial x'} - \frac{v^2}{c^3}\ \frac{\partial A'_x}{\partial t'}\right\} \tag{1.4-2}$$

Proceeding in the same way we find that

$$\frac{\partial A_x}{\partial x} = \beta^2\left\{\frac{\partial A_x}{\partial x'} - \frac{v}{c^2}\ \frac{\partial A'_x}{\partial t'} + \frac{v}{c}\ \frac{\partial\phi'}{\partial x'} - \frac{v^2}{c^3}\ \frac{\partial\phi'}{\partial t'}\right\} \tag{1.4-3}$$

$$\frac{\partial\phi}{\partial y} = \beta\left\{\frac{\partial\phi'}{\partial y'} + \frac{v}{c}\frac{\partial A'_x}{\partial y'}\right\} \qquad \frac{\partial A_x}{\partial y} = \beta\left\{\frac{\partial A'_x}{\partial y'} + \frac{v}{c}\frac{\partial\phi'}{\partial y'}\right\} \tag{1.4-4}$$

$$\frac{\partial A_y}{\partial x} = \beta\left\{\frac{\partial A'_y}{\partial x'} - \frac{v}{c^2}\frac{\partial A'_y}{\partial t'}\right\} \qquad \frac{\partial A_z}{\partial x} = \beta\left\{\frac{\partial A'_z}{\partial x'} - \frac{v}{c^2}\frac{\partial A'_z}{\partial t'}\right\} \tag{1.4-5}$$

$$\frac{\partial A_y}{\partial y} = \frac{\partial A'_y}{\partial y'} \qquad \frac{\partial A_z}{\partial y} = \frac{\partial A'_z}{\partial y'} \tag{1.4-6}$$

Derivatives with respect to z take the same form as those with respect to y.

Consider next the change of ϕ which accompanies the time increment Δt at O.

This produces a movement $-\bar{i}\beta v\Delta t$ and a time increment $\beta\Delta t$ at O'. The corresponding changes in ϕ' and A'_x are given by

$$\Delta\phi' \approx -\frac{\partial\phi'}{\partial x'}\beta v\Delta t + \frac{\partial\phi'}{\partial t'}\beta\Delta t$$

$$\Delta A'_x \approx -\frac{\partial A'_x}{\partial x'}\beta v\Delta t + \frac{\partial A'_x}{\partial t'}\beta\Delta t$$

or
$$\frac{\partial}{\partial t} = \beta\left\{\frac{\partial}{\partial t'} - v\frac{\partial}{\partial x'}\right\} \tag{1.4-7}$$

for operation upon a primed quantity.

Then
$$\frac{\partial\phi}{\partial t} = \beta^2\left\{\frac{\partial\phi'}{\partial t'} - v\frac{\partial\phi'}{\partial x'} + \frac{v}{c}\frac{\partial A'_x}{\partial t'} - \frac{v^2}{c}\frac{\partial A'_x}{\partial x'}\right\} \tag{1.4-8}$$

$$\frac{\partial A_x}{\partial t} = \beta^2\left\{\frac{\partial A'_x}{\partial t'} - v\frac{\partial A'_x}{\partial x'} + \frac{v}{c}\frac{\partial\phi'}{\partial t'} - \frac{v^2}{c}\frac{\partial\phi'}{\partial x'}\right\} \tag{1.4-9}$$

Further

$$\frac{\partial A_y}{\partial t} = \beta\left\{\frac{\partial A'_y}{\partial t'} - v\frac{\partial A'_y}{\partial x'}\right\} \qquad \frac{\partial A_z}{\partial t} = \beta\left\{\frac{\partial A'_z}{\partial t'} - v\frac{\partial A'_z}{\partial x'}\right\} \tag{1.4-10}$$

We are now in a position to derive the relationships between $\bar{E}$, $\bar{B}$ at 0 and $\bar{E}'$, $\bar{B}'$ at 0′. Thus we may utilise (1.4-2) and (1.4-9) to show that

$$\left\{-\frac{\partial\phi}{\partial x} - \frac{1}{c}\frac{\partial A_x}{\partial t}\right\}_0 = \left\{-\frac{\partial\phi'}{\partial x'} - \frac{1}{c}\frac{\partial A'_x}{\partial t'}\right\}_{0'},$$

or $E_x = E'_x$ at conjugate points.

The complete set is found to be

$$E_x = E'_x \qquad E_y = \beta(E'_y + \frac{v}{c}B'_z) \qquad E_z = \beta(E'_z - \frac{v}{c}B'_y)$$

$$B_x = B'_x \qquad B_y = \beta(B'_y - \frac{v}{c}E'_z) \qquad B_z = \beta(B'_z + \frac{v}{c}E'_y) \tag{1.4-11}$$

The reverse transformations require the interchange of primed and unprimed letters and the reversal of the sign of v.

As will be shown subsequently, these relationships also apply to certain macroscopic $\bar{E}$ and $\bar{B}$ fields, and it is in this context that they will be employed in later sections.

It may be remarked in passing that the transformations may be utilised to simplify the derivation of point source $\bar{E}$ and $\bar{B}$ fields in an S configuration when it is possible to bring one or more sources to rest (with zero acceleration) in S′. However, the requirement that at least one source in the primary configuration be acceleration-free severely limits the extent of the application (see Ex. 1-8).

EXERCISES

1-1. Confirm equations (1.2-5) to (1.2-7).

1-2. Derive the relationships (1.3-6) to (1.3-10).

1-3. Show that $(1-u'^2/c^2)^{1/2} = (1-u^2/c^2)^{1/2}/\beta(1-vu_x/c^2)$ where $u^2=u_x^2+u_y^2+u_z^2$.

Hence show that

$$\frac{u_x}{(1-u^2/c^2)^{1/2}},\ \frac{u_y}{(1-u^2/c^2)^{1/2}},\ \frac{u_z}{(1-u^2/c^2)^{1/2}},\ \frac{1}{(1-u^2/c^2)^{1/2}}$$

comprise the components of a four-vector (the velocity four-vector).

1-4. Use the equality in the previous exercise to demonstrate that $|u'|<c$ if $|u|<c$ and $|v|<c$.

1-5. If the masses of corresponding particles in the S and S′ configurations are given by

$$m = \frac{m_0}{(1-u^2/c^2)^{1/2}} \quad \text{and} \quad m' = \frac{m_0}{(1-u'^2/c^2)^{1/2}}$$

show that p_x, p_y, p_z, m comprise the components of a four-vector, where p is the momentum of the particle.

1-6. Show by expansion that $\bar{u}\cdot\frac{d\bar{p}}{dt} = m_0 u \frac{du}{dt}\Big/(1-u^2/c^2)^{3/2}$

Hence prove that

$$\frac{dp'_x}{dt'} = \frac{1}{(1-vu_x/c^2)}\left\{\frac{dp_x}{dt} - \frac{v}{c^2}\,\bar{u}\cdot\frac{d\bar{p}}{dt}\right\} = \frac{dp_x}{dt} - \frac{v/c^2}{(1-vu_x/c^2)}\left\{u_y\frac{dp_y}{dt} + u_z\frac{dp_z}{dt}\right\}$$

$$\frac{dp'_y}{dt'} = \frac{1}{\beta(1-vu_x/c^2)}\frac{dp_y}{dt} \;;\; \frac{dp'_z}{dt'} = \frac{1}{\beta(1-vu_x/c^2)}\frac{dp_z}{dt}$$

1-7. Confirm the relationships (1.4-11).

1-8. A point source of strength a moves in S with velocity $\bar{V}$ and zero acceleration. By transferring to an appropriate S′ configuration and back again, show that

$$\bar{E} = a\left(\frac{\bar{R}}{R} - \frac{\bar{V}}{c}\right)\frac{(1-V^2/c^2)}{\alpha^3 R^2}$$

$$\bar{B} = a\,(\bar{V}\times\bar{R})\,\frac{(1-V^2/c^2)}{\alpha^3 cR^3}$$

where R is the retarded distance between the source and point O of evaluation of $\bar{E}$ and $\bar{B}$, $\bar{R}$ is directed towards O, and $\alpha = (1-V_R/c)$.

Check the result by referring to equations (5.11-21/22), Pt. 1.

1.5 Transformation of Doublets and Whirls

In the preceding sections we have been concerned mainly with singlet sources which occupy their retarded positions in S relative to some designated point of evaluation of potential; and it has been shown that this particular relationship is retained subsequent to transformation.

Thus in Fig. 1.1 neither the S nor the S′ configuration represents conditions at a given instant. We now consider the manner in which extended source systems transform between fixed times in S and S′. For reasons which will become apparent later we will restrict considerations to doublets and whirls whose centres move in the x direction in S with a constant velocity which we identify with the arbitrary transformation constant v. The transformed centres are consequently at rest in S′.

1.5a Transformation of the doublet moment

Consider a dipole QP in the S configuration at the time t_1 as shown in Fig. 1.2.

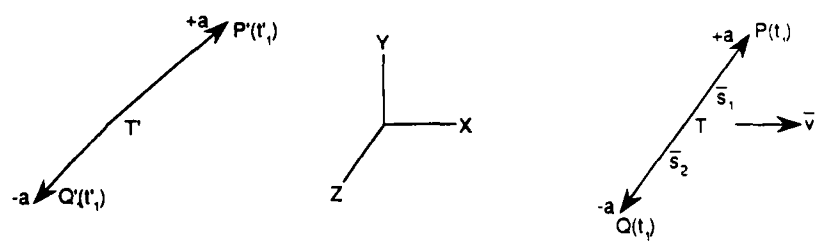

Fig. 1.2

The coordinates of the dipole centre T are taken to be $X+vt_1$, Y, Z, while $\overrightarrow{TP} = \bar{s}_1$ and $\overrightarrow{TQ} = \bar{s}_2$.

The dipole moment may be time-dependent through variation of singlet spacing or of orientation; in any case, the motion of the sources relative to T will be taken to be periodic.

The coordinates of T′ are βX, Y, Z while those of P′ are $\beta(X+s_{1x})$, $Y+s_{1y}$, $Z+s_{1z}$ at the time $\beta\left\{t_1 - \frac{v}{c^2}(X+vt_1+s_{1x})\right\}$.

Then at the time $\beta\left\{t_1 - \frac{v}{c^2}(X+vt_1)\right\}$ = say, t'_1, ie at the conjugate time deriving from T, the coordinates of P′ are given approximately by

$$\beta(X+s_{1x}) + \frac{\beta v}{c^2} s_{1x} u'_{1x} \; ; \; Y+s_{1y} + \frac{\beta v}{c^2} s_{1x} u'_{1y} \; ; \; Z+s_{1z} + \frac{\beta v}{c^2} s_{1x} u'_{1z}$$

where $\bar{u}'_1$ is the velocity of P′ at the time t'_1.

so that the components of $\overrightarrow{T'P'}$ at time t_1' are likewise

$$\beta s_{1x}(1+vu_{1x}'/c^2) \; ; \; s_{1y}+\frac{\beta v}{c^2} s_{1x}u_{1y}' \; ; \; s_{1z}+\frac{\beta v}{c^2} s_{1x}u_{1z}' \qquad (1.5\text{-}1)$$

Hence the contribution of the positive source of strength a at P′ to the polarisation relative to T′ has the components

$$\beta p_{1x}(1+vu_{1x}'/c^2) \qquad p_{1y}+\frac{\beta v}{c^2} p_{1x}\, u_{1y}' \qquad p_{1z}+\frac{\beta v}{c^2} p_{1x}\, u_{1z}'$$

where $\bar{p}_1 = a\bar{s}_1$

The contribution of the source of strength -a at Q′ is correspondingly

$$\beta p_{2x}(1+vu_{2x}'/c^2) \qquad p_{2y}+\frac{\beta v}{c^2} p_{2x}u_{2y}' \qquad p_{2z}+\frac{\beta v}{c^2} p_{2x}\, u_{2z}'$$

where $\bar{u}_2'$ is the velocity of Q′ at the time t_1' and $\bar{p}_2 = -a\bar{s}_2$.

If now PQ shrinks about T while a is increased to maintain $\bar{p}=\bar{p}_1+\bar{p}_2$ constant at any instant, the dipole reduces to a doublet and $\bar{u}_1'$ and $\bar{u}_2'$ approach zero. It follows that the relationship between the instantaneous moment $\bar{p}$ at the arbitrary time t_1 in S and the instantaneous conjugate moment $\bar{p}'$ in S′ is given by

$$p_x' = \beta p_x \qquad p_y' = p_y \qquad p_z' = p_z \qquad (1.5\text{-}2)$$

1.5b <u>Transformation of the whirl moment</u>

Let a closed curve Γ translate in S with a uniform velocity $\bar{i}v$. Then Γ transforms to a closed curve Γ′, stationary in S′.

Suppose that closely-spaced positive singlets of equal magnitude are uniformly distributed around Γ and move with constant speed w relative to it in an anticlockwise direction (Fig. 1.3). An equal number of negative singlets of the same magnitude are similarly distributed around Γ and are stationary relative to it. Then the current in Γ is time-invariant and equal at all points, and is given by $\lambda^{+}w$ where λ^{+} is the positive source strength per unit length, while the net source density is everywhere zero. Since each source always coincides with some point of Γ the transformed sources will always lie upon Γ′.

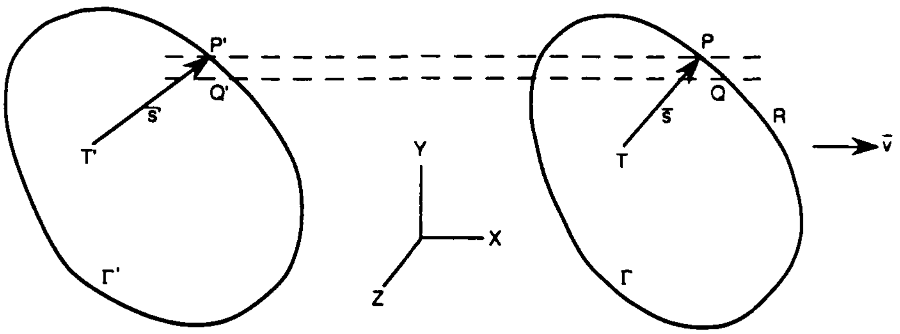

Fig. 1.3

Consider the transformed positive source located at $P'(x',y',z')$ at the time t'. This will have derived from a parent source at P in Γ having the coordinates $\beta(x'+vt')$, y',z' at the time $\beta(t'+vx'/c^2)$. Let the following source arrive at P' at the time $t'+\Delta t'$. This will have derived from a source located at P [now with the coordinates $\beta(x'+ v(t'+ \Delta t')),y',z'$] at the time $\beta(t'+\Delta t'+vx'/c^2)$. Hence the passage of consecutive conjugate sources at P' and P occupy the intervals $\Delta t'$ and $\beta\Delta t'$. Since the source strengths are unchanged upon transformation it follows that

$$I' = \beta I \qquad (1.5\text{-}3)$$

This relationship will apply generally since I is constant around Γ and time-invariant. Since, in addition, stationary negative sources in Γ transform to stationary sources in Γ', there can be no time variation of source density around Γ'. However, as will now be shown, a time-invariant net source density develops in Γ'.

As stated above, a positive source at P' at time t' derives from a source at P at time $\beta(t'+vx'/c^2)$. Similarly a positive source at $Q'(x'+\Delta x', y'+\Delta y',z'+\Delta z')$ at the time t' derives from a source located at Q at the time $\beta\left(t'+ \frac{v}{c^2}(x'+\Delta x')\right)$. Now at the time $\beta(t'+vx'/c^2)$ the latter source will not have reached Q but will be at a distance $\frac{\beta v}{c^2}\Delta x'w$ back along the contour - say at R. All of the positive sources between P and R at the time $\beta(t'+vx'/c^2)$ in S will transform into sources between P' and Q' at the time t' in S'. Hence the total positive source strength between P' and Q' at the time t' is given by $\lambda^{+}\left\{PQ+ \frac{\beta v}{c^2}\Delta x'w\right\}$. The negative source strength between P' and Q' is simply $-\lambda^{+}PQ$ hence the net source strength between P' and Q' at time t' is

$$\frac{\beta v I \Delta x'}{c^2} = - \frac{v I' \ \Delta \bar{s}' \cdot \bar{1}}{c^2} = - \frac{v}{c^2} \bar{I}' \cdot \bar{1} |\Delta \bar{s}'|$$

where $\bar{s}'$ is the position vector of P′ relative to a local origin T′, the positive sense of $\Delta \bar{s}'$ is anticlockwise, and $\bar{I}' |\Delta \bar{s}'| = I' \Delta \bar{s}'$.

Thus the net source strength per unit length at P′ is given by

$$\lambda' = - \frac{v}{c^2} \bar{I}' \cdot \bar{1} \qquad (1.5\text{-}4)$$

Since I is time-invariant and constant around Γ, and t′ can have any value, equation (1.5-4) holds for each point of Γ′ at all times. The polarisation of Γ′ relative to T′ is then given by

$$\bar{p}' = \oint_{\Gamma'} \bar{s}' \lambda' |d\bar{s}'| = - \frac{v}{c^2} I' \oint_{\Gamma'} \bar{s}' \ \bar{1} \cdot d\bar{s}'$$

But

$$d(\bar{1} \cdot \bar{s}' \bar{s}') = \bar{1} \cdot \bar{s}' \ d\bar{s}' + \bar{1} \cdot d\bar{s}' \bar{s}'$$

hence

$$\oint_{\Gamma'} \bar{1} \cdot \bar{s}' d\bar{s}' = - \oint_{\Gamma'} \bar{s}' \bar{1} \cdot d\bar{s}'$$

and

$$\oint_{\Gamma'} (\bar{1} \cdot \bar{s}' d\bar{s}' - \bar{s}' \bar{1} \cdot d\bar{s}') = -2 \oint_{\Gamma'} \bar{s}' \bar{1} \cdot d\bar{s}'$$

or

$$\oint_{\Gamma'} \bar{s}' \bar{1} \cdot d\bar{s}' = 1/2 \oint_{\Gamma'} \bar{1} \times (\bar{s}' \times d\bar{s}') = \bar{1} \times \bar{S}' \qquad (1.5\text{-}5)$$

where $\bar{S}'$ is the area of a simple surface[2] spanning Γ′,

so that for a time-invariant value of I,

$$\bar{p}' = -\beta \frac{v}{c^2} I (\bar{1} \times \bar{S}') \qquad (1.5\text{-}6)$$

A re-tracing of the steps leading to (1.5-3) and (1.5-4) reveals that these relationships continue to hold at conjugate points when I is time-dependent. However, I′ will no longer be constant around Γ′ at a given instant in S′, although I is constant around Γ.

---

2. There is no cause to confuse the vector area $\bar{S}'$ with the configuration indicator S′.

Since

$$I'_{x',y',z',t'} = \beta I_{\Gamma,\ \beta(t'+vx'/c^2)}$$

we have

$$I'_{x',y',z',t'} = \beta\left\{I + \frac{dI}{dt}\,\frac{\beta v}{c^2}\,(x'-x'_1) + \ldots\ldots\right\} \tag{1.5-7}$$

where I and $\frac{dI}{dt}$ are evaluated in Γ at the particular time $t = \beta(t'+vx'_1/c^2)$, the coordinates of T' being x'_1, y'_1, z'_1.

In this case the polarisation of Γ' relative to T' becomes

$$\bar{p}'_{t'} = \oint_{\Gamma',t'} \bar{s}'\lambda'|d\bar{s}'| = \frac{-\beta v}{c^2}\left\{I_t\oint_{\Gamma'}\bar{s}'\,\bar{\imath}\cdot d\bar{s}' + \frac{\beta v}{c^2}\left(\frac{dI}{dt}\right)_t\oint_{\Gamma'}\bar{s}'\,\bar{\imath}\cdot\bar{s}'\,\bar{\imath}\cdot d\bar{s}' + \ldots\ldots\right\}$$

$$= -\frac{\beta v}{c^2}\,I_t(\bar{\imath}\times\bar{S}') - \frac{\beta^2v^2}{c^4}\left(\frac{dI}{dt}\right)_t\oint_{\Gamma'}\bar{s}'\,\bar{\imath}\cdot\bar{s}'\,\bar{\imath}\cdot d\bar{s}' + \ldots\ldots \tag{1.5-8}$$

The current moment of Γ' about T' is given by[3]

$$\bar{m}'_{t'} = \frac{1}{2c}\oint_{\Gamma'}\bar{s}'\times I'_{t'}\,d\bar{s}'$$

$$= \frac{1}{2c}\oint_{\Gamma'}\bar{s}'\times\beta\left\{I_t + \frac{\beta v}{c^2}\left(\frac{dI}{dt}\right)_t(x'-x'_1) + \ldots\ldots\ldots\right\}d\bar{s}'$$

$$= \beta I_t\,\frac{\bar{S}'}{c} + \frac{\beta^2 v}{2c^3}\left(\frac{dI}{dt}\right)_t\oint_{\Gamma'}\bar{s}'\times s'_x\,d\bar{s}' + \ldots\ldots\ldots \tag{1.5-9}$$

Suppose now that Γ shrinks uniformly about T while I is increased to maintain $I\bar{S}$ constant at any moment ($\bar{S}$ being the vector area bounded by Γ). Then Γ' shrinks uniformly about T' while $I\bar{S}'$ remains constant. The second and higher-order terms of the series in (1.5-8) and (1.5-9) vanish on dimensional grounds, so that the polarisation and current moment of the resulting whirl are given by

$$\bar{p}'_{t'} = -\frac{v}{c}\left\{\bar{\imath}\times \mathrm{Lim}\,\beta I_t\bar{S}'/c\right\} \tag{1.5-10}$$

and

3. See footnotes to pp. 381 and 519, Pt. 1.

$$\bar{m}'_{t'} = \text{Lim } \beta I_t \bar{S}'/c \qquad (1.5\text{-}11)$$

whence

$$\bar{p}' = -\frac{v}{c}(\bar{i} \times \bar{m}') \qquad (1.5\text{-}12)$$

It is easily shown that the x dimensions of Γ' are β times those of Γ ; y and z dimensions are, of course, unchanged. Then

$$\bar{i}S'_x + \bar{j}S'_y + \bar{k}S'_z = \bar{i}S_x + \bar{j}\beta S_y + \bar{k}\beta S_z \qquad (1.5\text{-}13)$$

Since $\bar{m}_t = \text{Lim } I_t\bar{S}/c$, substitution in (1.5-11) yields the following conjugate relationships

$$m'_x = \beta m_x \qquad m'_y = \beta^2 m_y \qquad m'_z = \beta^2 m_z \qquad (1.5\text{-}14)$$

Then from (1.5-12)

$$p'_x = 0 \qquad p'_y = \frac{v}{c}\, m'_z = \beta^2 \frac{v}{c}\, m_z \qquad p'_z = -\frac{v}{c}\, m'_y = -\beta^2 \frac{v}{c}\, m_y \qquad (1.5\text{-}15)$$

The above analysis addresses only motion of translation in S. If the contour should, in addition, be subject to rotation about an axis through T, as when time-dependence of whirl moment is due to variation of orientation rather than of intrinsic magnitude, the behaviour under transformation is more complicated. Thus although current may be time-invariant and constant around Γ, the current in Γ' will be a function of position around the contour, and the simple areal relationships no longer apply. Nevertheless, it may be shown that equations (1.5-14) and (1.5-15) remain valid. The relevant analysis, together with that relating to the determination of the corresponding retarded potentials, is the subject of Ex. 1-11 and Ex. 1-15/18.

EXERCISES

1-9. Derive equation (1.5-4) for the time-dependent case by utilising the conservation equation $\dfrac{\partial\lambda'}{\partial t'} = -\dfrac{\partial I'}{\partial|\bar{s}'|}$

1-10. By substitution of (1.5-7) in (1.5-4) and subsequent contour integration show that the total source strength in Γ' is zero when Γ comprises a translating contour carrying a uniform current. (The polarisation of the whirl may consequently be described as a doublet moment).

Confirm this by demonstrating that if $|v|<c$ and $|w|<c$ there is a one to one relationship between sources in Γ and those in Γ' at a given instant in Γ'.

Utilise the value of λ' given in Ex. 1-11 to show that the above result continues to hold when Γ comprises a translating, rotating contour carrying a time-invariant current.

1-11. A closed contour Γ rotates in S about an axis through T (Fig. 1.3) while the axis translates with velocity $\bar{i}v$. The transformed contour Γ' consequently rotates about a fixed axis in S'. Γ carries a time-invariant, neutral current I.

Show that the instantaneous source density in Γ' is given by $\lambda' = -\beta I\ \bar{v}\cdot\hat{\bar{t}}'/c^2$ where $\hat{\bar{t}}'$ is the unit tangent to Γ' in the direction of the current, and that $I' = \beta I(1 + \bar{v}\cdot\bar{u}'/c^2)$ where $\bar{u}'$ is the velocity of Γ' at the point under consideration.

Then show that equations (1.5-10) to (1.5-15) continue to hold in the limiting case.

1-12. It was shown in Secs. 5.14/15, Pt. 1, that the scalar and vector potentials of a point whirl could be rendered constant in time by an appropriate arrangement of point sources around a circular orbit. This arrangement made it possible to handle time-dependent whirl moments without the restrictions imposed by time-averaging. In the present chapter it has been necessary to employ a quasi-continuous whirl to overcome this difficulty.

It is of interest to note that the transformation equations (1.5-14/15) remain valid for a whirl comprising one or more point sources moving in one or more closed curves of arbitrary shape, provided that the expressions are understood to represent time-averages.

Suppose that a point source of strength a moves in a closed curve with relative velocity $\bar{w}$ about a point T which translates in S with velocity $\bar{i}v$ (Fig. 1-3). If the associated polarisation and current moment in S are defined as the time-averages of

$$\bar{p} = a\bar{s} \qquad \text{and} \qquad \bar{m} = \frac{1}{2c}(\bar{s} \times a\bar{w})$$

with corresponding expressions in S', show that

$$p'_x = \beta p_x \qquad p'_y = p_y + \beta^2 \frac{v}{c} m_z \qquad p'_z = p_z - \beta^2 \frac{v}{c} m_y$$

$$m'_x = \beta m_x \qquad m'_y = \beta^2 m_y \qquad m'_z = \beta^2 m_z$$

[Hint: First show that (period in S') = (period in S)/β, and

$$\Delta t' = \beta\left\{1 - v(w_x + v)/c^2\right\} \Delta t]$$

(1.5-15) differs from the above expression for $\bar{p}'$ by the omission of $\bar{p}$ components. Show that it would assume the above form if the quasi-continuous whirl possessed a polarisation $\bar{p}$ in S.

1.6 The Retarded Potentials of Transformed Doublets and Whirls

It is to be expected that when transformed doublets and whirls have stationary centres in S′ the expressions for their scalar and vector potentials will take the same form as those developed in Part 1, in which case the values of the transformed doublet and whirl moments may be substituted directly in such formulae. However, the limiting procedure may now be complicated by the accompanying variation of source configuration occasioned by the transformation itself, and it is consequently useful to confirm that any additional terms engendered in this way will vanish in the limit.

1.6a Scalar and vector potentials of a transformed doublet

The required S′ configuration is depicted in Fig.1.4. O′ is the point at which the retarded potentials of the transformed sources are to be determined at the time t′. T′ is stationary, being the transformed position of the dipole centre T in S which moves with velocity $\bar{i}v$. P′ and Q′ are the positions of the sources in S′ at the time $t'-R'/c$ where $\bar{R}' = \overrightarrow{T'O'}$, while $\overrightarrow{T'P'}$ and $\overrightarrow{T'Q'}$ are represented by $\bar{s}_1'$ and $\bar{s}_2'$ respectively. (When the dipole moment in S is time-dependent, T′ will not, in general, bisect P′Q′, nor will the points P′,T′,Q′ be collinear).

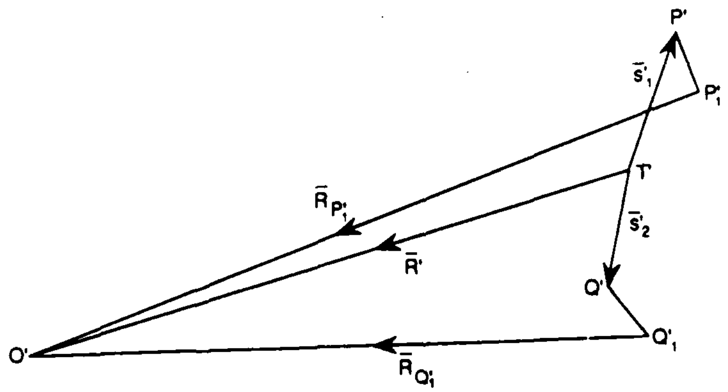

Fig. 1.4

If P_1', and Q_1', are the retarded positions of +a and -a appropriate to the required evaluation, the scalar potential at O′ is given by

$$\phi' = a\Big/R_{P_1'}\left\{1 - \bar{u}'_{P_1'}\cdot\bar{R}_{P_1'}/cR_{P_1'}\right\} - a\Big/R_{Q_1'}\left\{1 - \bar{u}'_{Q_1'}\cdot\bar{R}_{Q_1'}/cR_{Q_1'}\right\} \qquad (1.6\text{-}1)$$

where $\bar{u}'$ denotes source velocity.

The evaluation of the limiting form of this type of expression has been carried out in Sec. 5.13, Pt. 1, where it was supposed that the dipole was subject to rotation while its magnitude varied sinusoidally with time. We may proceed along slightly more general lines by simply supposing that the dipole moment is continuous and periodic in time, in which case u' may be taken to be of the same order as s'. We then find that to a first order in s'

$$\phi' = \frac{a}{R'}\left\{1 + \frac{\bar{R}'\cdot\bar{s}'_1}{R'^2} + \frac{\bar{R}'\cdot\bar{u}'_{P'}}{cR'}\right\} - \frac{a}{R'}\left\{1 + \frac{\bar{R}'\cdot\bar{s}'_2}{R'^2} + \frac{\bar{R}'\cdot\bar{u}'_{Q'}}{cR'}\right\}$$

or

$$\phi' = \frac{\bar{R}'}{R'^3}\cdot a(\bar{s}'_1 - \bar{s}'_2) + \frac{\bar{R}'}{cR'^2}\cdot\frac{d}{dt'}\, a(\bar{s}'_1 - \bar{s}'_2)$$

Now from (1.5-1) we see that to a first order in s'

$$s'_x = \beta s_x \qquad s'_y = s_y \qquad s'_z = s_z$$

hence, in the limit, as the dipole reduces to a doublet and the second and higher-order terms in s' vanish, we are left with the not-unexpected result

$$\phi' = \frac{\bar{R}'}{R'^3}\cdot[\bar{p}'] + \frac{\bar{R}'}{cR'^2}\cdot\left[\frac{d\bar{p}'}{dt'}\right] \tag{1.6-2}$$

where $\bar{p}'$ is defined by (1.5-2) and [ ] implies evaluation at the time $t'-R'/c$.

The corresponding vector potential is found by evaluating

$$\bar{A}' = a\bar{u}'_{P'_1}\Big/cR_{P'_1}\left\{1 - \bar{u}'_{P'_1}\cdot\bar{R}'_{P'_1}\Big/cR_{P'_1}\right\} - a\bar{u}'_{Q'_1}\Big/cR_{Q'_1}\left\{1 - \bar{u}'_{Q'_1}\cdot\bar{R}_{Q'_1}\Big/cR_{Q'_1}\right\}$$

which, to a first order in s', reduces to

$$\bar{A}' = a\bar{u}'_{P'}\big/cR' - a\bar{u}'_{Q'}\big/cR' = \frac{1}{cR'}\frac{d}{dt'}\, a(\bar{s}'_1 - \bar{s}'_2)$$

whence, in the limit,

$$\bar{A}' = \frac{1}{cR'}\left[\frac{d\bar{p}'}{dt'}\right] \tag{1.6-3}$$

1.6b Scalar potential of a transformed whirl

It will be supposed that the whirl translates in S with velocity $\bar{i}v$ and is time-dependent in virtue of current variation.

The contribution of the stationary contour element $|d\bar{s}'|$ at P' (as in Fig. 1.3) to the retarded scalar potential of Γ' at the point 0' at the time t' is given by

$$d\phi' = [\lambda']_{P'} \; |d\bar{s}'| \Big/ R_{P'} \tag{1.6-4}$$

where $[\lambda']_{P'}$ is the instantaneous net linear source density at P' at the time $t'-R_{P'}/c$, and $R_{P'}=P'0'$.

[It will be recalled that the Liénard factor in the denominator of the expression for the potential of an individual singlet source (Sec. 5.10, Pt. 1), is cancelled by the factor relating the instantaneous line or volume density to the density of individually-retarded sources (Sec. 5.17)]

Then

$$d\phi' = \left\{\frac{1}{R'}+\frac{\bar{R}'\cdot\bar{s}'}{R'^3}\right\} |d\bar{s}'| \; \lambda'_{P',t'-R_{P'}/c} \tag{1.6-5}$$

where $\bar{R}' = \overrightarrow{T'0'}$, and the terms in s'^2 and higher orders have been ommitted.

Since I is time-dependent in S, λ' will be time-dependent in S', and, to the first order in s',

$$\lambda'_{P',t'-R_{P'}/c} = \left\{\lambda' + \frac{d\lambda'}{dt'}\,\frac{\bar{R}'\cdot\bar{s}'}{cR'}\right\}_{P',t'-R'/c}$$

whence, from (1.5-4),

$$|d\bar{s}'| \; \lambda'_{P',t'-R_{P'}/c} = -\frac{v}{c^2}\,\bar{i}\cdot d\bar{s}' \left\{I' + \frac{dI'}{dt'}\,\frac{\bar{R}'\cdot\bar{s}'}{cR'}\right\}_{P',t'-R'/c} \tag{1.6-6}$$

On taking the coordinates of P' and T' as x',y',z' and x_1',y_1',z_1' respect-respectively, we have

$$I'_{P',t'-R'/c} = \beta I_{\beta(t'-\frac{R'}{c}+\frac{v}{c^2}x')} = \beta\left\{I + \frac{dI}{dt}\,\frac{\beta v}{c^2}\,\bar{i}\cdot\bar{s}'+\ldots\right\}_{\beta(t'-\frac{R'}{c}+\frac{v}{c^2}x_1')}$$

Further, since $\frac{d}{dt'} = \beta \frac{d}{dt}$,

$$\left(\frac{dI'}{dt'}\right)_{P',t'-R'/c} = \left\{\beta^2 \frac{dI}{dt} + \ldots\right\}_{\beta(t' - \frac{R'}{c} + \frac{v}{c^2} x_1')}$$

On substituting for I' and $\frac{dI'}{dt'}$ in (1.6-6) and for λ' in (1.6-5), and writing $\beta(t' - \frac{R'}{c} + \frac{v}{c^2} x_1')$ as t_1, we find that to a first order in s'

$$d\phi' = -\frac{v}{c^2}\,\bar{1}\cdot d\bar{s}' \left\{\frac{\beta}{R'} I_{t_1} + \beta \frac{\bar{R}'\cdot\bar{s}'}{R'^3} I_{t_1} + \frac{\beta^2 v}{c^2 R'}\,\bar{1}\cdot\bar{s}'\left(\frac{dI}{dt}\right)_{t_1} + \beta^2 \frac{\bar{R}'\cdot\bar{s}'}{cR'^2}\left(\frac{dI}{dt}\right)_{t_1}\right\}$$

The first and third terms vanish upon integration, leaving

$$\phi' = -\frac{v}{c^2}\left\{\beta I_{t_1} \frac{\bar{R}'}{R'^3}\cdot \oint_{\Gamma'} \bar{s}'\,\bar{1}\cdot d\bar{s}' + \beta^2\left(\frac{dI}{dt}\right)_{t_1} \frac{\bar{R}'}{cR'^2}\cdot \oint_{\Gamma'} \bar{s}'\,\bar{1}\cdot d\bar{s}'\right\}$$

$$= -\frac{v}{c^2}\left\{\beta I_{t_1} \frac{\bar{R}'}{R'^3}\cdot(\bar{1}\times\bar{S}') + \beta^2\left(\frac{dI}{dt}\right)_{t_1} \frac{\bar{R}'}{cR'^2}\cdot(\bar{1}\times\bar{S}')\right\}$$

Then in the limit as Γ shrinks about T and Γ' about T', and as I is increased to maintain $I\,\bar{S}$ constant at any instant, the scalar potential of the transformed whirl is seen to be given by

$$\phi' = -\frac{v}{c}\frac{\bar{R}'}{R'^3}\cdot\left\{\bar{1}\times \mathrm{Lim}\,\beta\frac{\bar{S}'}{c} I_{t_1}\right\} - \frac{v}{c}\frac{\bar{R}'}{cR'^2}\cdot\left\{\bar{1}\times \mathrm{Lim}\,\beta^2\frac{\bar{S}'}{c}\left(\frac{dI}{dt}\right)_{t_1}\right\}$$

But from (1.5-11), $\bar{m}'_{t'} = \mathrm{Lim}\,\beta\frac{\bar{S}'}{c} I_{\beta(t' + vx_1'/c^2)}$

hence $\bar{m}'_{t' - R'/c} = \mathrm{Lim}\,\beta\frac{\bar{S}'}{c} I_{t_1}$ and $\left(\frac{d\bar{m}'}{dt'}\right)_{t' - R'/c} = \mathrm{Lim}\,\beta^2\frac{\bar{S}'}{c}\left(\frac{dI}{dt}\right)_{t_1}$

so that

$$\phi' = -\frac{v}{c}\frac{\bar{R}'}{R'^3}\cdot\left\{\bar{1}\times\bar{m}'_{t' - R'/c}\right\} - \frac{v}{c}\frac{\bar{R}'}{cR'^2}\cdot\left\{\bar{1}\times\left(\frac{d\bar{m}'}{dt'}\right)_{t' - R'/c}\right\}$$

or

$$\phi' = \frac{\bar{R}'}{R'^3}\cdot[\bar{p}'] + \frac{\bar{R}'}{cR'^2}\cdot\left[\frac{d\bar{p}'}{dt'}\right] \qquad (1.6\text{-}7)$$

where $\bar{p}'$ is defined by (1.5-12).

1.6c Vector potential of a transformed whirl

In the notation of the previous subsection, the contribution of the current element $d\bar{s}'$ at P′ to the vector potential at O′ at time t′ may be written as

$$d\bar{A}' = [I']_{P'}\ d\bar{s}'/cR_{P'} \tag{1.6-8}$$

so that to a first order in s′

$$d\bar{A}' = \frac{1}{c}\, d\bar{s}' \left\{\frac{1}{R'} + \frac{\bar{R}'\cdot\bar{s}'}{R'^3}\right\}\left\{I' + \frac{dI'}{dt'}\,\frac{\bar{R}'\cdot\bar{s}'}{cR'}\right\}_{P',\,t'-R'/c}$$

$$= \frac{1}{c}\, d\bar{s}' \left\{\frac{I'}{R'} + I'\,\frac{\bar{R}'\cdot\bar{s}'}{R'^3} + \frac{dI'}{dt'}\,\frac{\bar{R}'\cdot\bar{s}'}{cR'^2}\right\}_{P',\,t'-R'/c}$$

whence

$$d\bar{A}' = \frac{1}{c}\, d\bar{s}'\left\{\frac{\beta}{R'}\, I_{t_1} + \beta\,\frac{\bar{R}'\cdot\bar{s}'}{R'^3}\, I_{t_1} + \beta^2\,\frac{\bar{R}'\cdot\bar{s}'}{cR'^2}\left(\frac{dI}{dt}\right)_{t_1} + \frac{\beta^2 v}{c^2R'}\,\bar{1}\cdot\bar{s}'\left(\frac{dI}{dt}\right)_{t_1}\right\}$$

The first term vanishes upon integration, leaving

$$\bar{A}' = \frac{1}{c}\left\{\beta I_{t_1}\oint_{\Gamma'}\frac{\bar{R}'}{R'^3}\cdot\bar{s}'\ d\bar{s}' + \beta^2\left(\frac{dI}{dt}\right)_{t_1}\oint_{\Gamma'}\frac{\bar{R}'}{cR'^2}\cdot\bar{s}'\ d\bar{s}' + \frac{\beta^2 v}{c^2R'}\left(\frac{dI}{dt}\right)_{t_1}\oint_{\Gamma'}\bar{1}\cdot\bar{s}'\ d\bar{s}'\right\}$$

It is easily shown that $\oint_{\Gamma'} \bar{a}\cdot\bar{s}'\ d\bar{s}' = -\,\bar{a}\times\bar{S}'$ where $\bar{a}$ is a constant vector, hence

$$\bar{A}' = -\,\frac{1}{c}\left\{\beta I_{t_1}\,\frac{\bar{R}'}{R'^3}\times\bar{S}' + \beta^2\left(\frac{dI}{dt}\right)_{t_1}\frac{\bar{R}'}{cR'^2}\times\bar{S}' + \frac{v}{c^2R'}\,\beta^2\left(\frac{dI}{dt}\right)_{t_1}(\bar{1}\times\bar{S}')\right\}$$

Then in the limiting case, in which the suppressed terms vanish,

$$\bar{A}' = \left\{\mathrm{Lim}\ \beta\,\frac{\bar{S}'}{c}\, I_{t_1}\right\}\times\frac{\bar{R}'}{R'^3} + \left\{\mathrm{Lim}\ \beta^2\,\frac{\bar{S}'}{c}\left(\frac{dI}{dt}\right)_{t_1}\right\}\times\frac{\bar{R}'}{cR'^2} + \left\{\mathrm{Lim}\ \beta^2\,\frac{\bar{S}'}{c}\left(\frac{dI}{dt}\right)_{t_1}\right\}\times\frac{\bar{1}v}{c^2R'}$$

whence, from previous considerations,

$$\bar{A}' = [\bar{m}'] \times \frac{\bar{R}'}{R'^3} + \left[\frac{d\bar{m}'}{dt'}\right] \times \frac{\bar{R}'}{cR'^2} + \frac{1}{cR'}\left[\frac{d\bar{p}'}{dt'}\right] \tag{1.6-9}$$

where $\bar{p}'$ is defined by (1.5-12).

EXERCISES

1-13. The centre T of a dipole moves with uniform velocity $\bar{u}$. The dipole moment is time-dependent in virtue of rotation and/or variation of singlet spacing. At the time t-R/c, T is located at a distance R from O while the sources +a and -a occupy the positions P and Q respectively. If P_1 and Q_1 are the appropriately-retarded positions of +a and -a for potential evaluation at O at time t, then

$$\phi = \frac{a}{R_{P_1}\left\{1 - (\bar{u} + \bar{w}_{P_1})\cdot\bar{R}_{P_1}/cR_{P_1}\right\}} - \frac{a}{R_{Q_1}\left\{1 - (\bar{u} + \bar{w}_{Q_1})\cdot\bar{R}_{Q_1}/cR_{Q_1}\right\}}$$

where $\bar{w}$ is the source velocity relative to T and $\bar{R}$ is directed towards O.

Expand this in the form

$$\phi = \frac{a}{R\left\{1-(\bar{u} + \bar{w}_P)\cdot\bar{R}/cR\right\}} + \Delta\phi^+ - \frac{a}{R\left\{1-(\bar{u} + \bar{w}_Q)\cdot\bar{R}/cR\right\}} - \Delta\phi^-$$

and by taking limits show that the associated scalar doublet potential is given by

$$\phi = \frac{\bar{R}}{R^3}\cdot[\bar{p}]\,\frac{(1-u^2/c^2)}{\alpha^3} + \frac{\bar{R}}{cR^2}\cdot\left[\frac{d\bar{p}}{dt}\right]\frac{1}{\alpha^2} - \frac{[\bar{p}]\cdot\bar{u}}{\alpha^2 cR^2}$$

where $\alpha = 1 - u_R/c$.

Show further that the corresponding vector potential is given by

$$\bar{A} = \frac{1}{\alpha cR}\left[\frac{d\bar{p}}{dt}\right] + \frac{\bar{u}}{c}\,\frac{\bar{R}}{R^3}\cdot[\bar{p}]\,\frac{(1-u^2/c^2)}{\alpha^3} + \frac{\bar{u}}{c}\,\frac{\bar{R}}{cR^2}\cdot\left[\frac{d\bar{p}}{dt}\right]\frac{1}{\alpha^2} - \frac{\bar{u}}{c}\,\frac{[\bar{p}]\cdot\bar{u}}{\alpha^2 cR^2}$$

Observe the complexity of these expressions compared with those for which $\bar{u} = 0$.

1.14. Derive the results of the previous exercise for the case $\bar{u} = \bar{i}u$ by bringing the doublet centre to rest in an S′ configuration and transforming the associated scalar and vector potentials, viz

$$\phi' = \frac{\bar{R}'}{R'^3}\cdot[\bar{p}'] + \frac{\bar{R}'}{cR'^2}\cdot\left[\frac{d\bar{p}'}{dt'}\right] \qquad \bar{A}' = \frac{1}{cR'}\left[\frac{d\bar{p}'}{dt'}\right]$$

back into ϕ and $\bar{A}$, via equations (1.3-1/4), (1.3-9/10) and (1.5-2).

1-15. At the time t-R/c a contour element $d\bar{s}$ is located at a distance R from a point O. The element carries a current I and a net source density λ, and it moves as a whole with velocity $\bar{u}$. Show that its contribution to the vector potential at O at the time t is given by

$$d\bar{A} = \frac{\lambda\bar{u}\,|d\bar{s}|}{cR(1 - u_R/c)} + \frac{I\,d\bar{s}}{cR(1 - u_R/c)}$$

where $\bar{R}$ is directed towards O.

[Proceed by evaluating the separate contributions of positive and negative singlets (supposing that the latter are at rest in $d\bar{s}$) through a determination of the effective individually-retarded densities, taking due account of the Liénard factors.]

1.16. Show that

(1) $\frac{d}{dt}(d\bar{s}) = d\bar{u}$

where $d\bar{u}$ is the change of velocity $\bar{u}$ between the end points of $d\bar{s}$.

(2) $\frac{d\bar{S}}{dt} = \oint_\Gamma \bar{u}\times d\bar{s} = \oint_\Gamma \bar{s}\times d\bar{u}$

where $\bar{S}$ is the vector area defined by Γ.

1-17. Proceed in the following way to demonstrate that the retarded vector potential of the transformed whirl of Ex.1-11 may be expressed as

$$[\bar{m}']\times\frac{\bar{R}'}{R'^3} + \left[\frac{d\bar{m}'}{dt'}\right]\times\frac{\bar{R}'}{cR'^2} + \frac{1}{cR'}\left[\frac{d\bar{p}'}{dt'}\right]$$

$$\text{where } \bar{m}' = \text{Lim } \beta I\frac{\bar{S}'}{c}$$

First show that the component of the vector potential deriving from the transport of λ' (Ex. 1-15) is given, to a first order in s', by $-\frac{\beta I}{c^3 R'} \oint_{\Gamma'} \bar{u}'\ \bar{v}\cdot d\bar{s}'$.

Then show that the component deriving from I' is given by

$$\frac{\beta I}{c} \oint_{\Gamma'} \left\{\frac{1}{R'} + \frac{\bar{R}'\cdot\bar{s}'}{R'^3} + \frac{\bar{R}'\cdot\bar{u}'}{cR'^2} + \ldots\right\} \left\{d\bar{s}' + d\bar{u}'\ \frac{\bar{R}'\cdot\bar{s}'}{cR'} + \ldots\right\} \left\{1 + \frac{\bar{v}\cdot\bar{u}'}{c^2}\right\}$$

and reduce this to the first-order expression

$$\frac{\beta I}{c} \oint_{\Gamma'} \left\{\frac{\bar{R}'\cdot\bar{s}'}{R'^3}\, d\bar{s}' + \frac{\bar{R}'\cdot\bar{u}'}{cR'^2}\, d\bar{s}' + \frac{\bar{R}'\cdot\bar{s}'}{cR'^2} d\bar{u}' + \frac{\bar{v}\cdot\bar{u}'}{c^2 R'}\, d\bar{s}'\right\}$$

Transform this into

$$\frac{\beta I}{c} \oint_{\Gamma'} \frac{\bar{R}'\cdot\bar{s}'}{R'^3}\, d\bar{s}' - \frac{\beta I}{c} \oint_{\Gamma'} \frac{\bar{R}'}{cR'^2} \times (\bar{u}' \times d\bar{s}') + \frac{\beta I}{c} \oint_{\Gamma'} \frac{\bar{v}\cdot\bar{u}'}{c^2 R'}\, d\bar{s}'$$

and, by combining the last term with the component of the vector potential deriving from λ', making appropriate transformations and taking limits, arrive at the required result.

1-18. Follow a procedure similar to that employed in the previous exercise to show that the retarded scalar potential of the same transformed whirl is given by

$$\phi' = \frac{\bar{R}'}{R'^3}\cdot\left[\bar{p}'\right] + \frac{\bar{R}'}{cR'^2}\cdot\left[\frac{d\bar{p}'}{dt'}\right]$$

$$\text{where } \bar{p}' = -\frac{v}{c}\,(\bar{i} \times \bar{m}')$$

1.7 Transformation of Population Density

We may determine the manner in which the instantaneous population density of a source distribution changes between conjugate points by applying the Lorentz transformation to the points of a closed surface which is immersed in the distribution and moves with it, and subsequently calculating the change of enclosed volume.

Consider a straight line AB which is undergoing uniform motion of translation in S with component velocities u_x, u_y, u_z. Let the coordinates of A and B at time t be x_1, y_1, z_1 and x_2, y_2, z_2. Then A transforms to the

point $\beta(x_1-vt), y_1, z_1$ at the time $\beta(t-vx_1/c^2)$= say, t', and B transforms to the point $\beta(x_2-vt), y_2, z_2$ at the time $\beta(t-vx_2/c^2)$. Then at the time t' the coordinates of B' will be

$$\beta(x_2-vt) + \frac{\beta v}{c^2}(x_2-x_1)u'_x \quad ; \quad y_2 + \frac{\beta v}{c^2}(x_2-x_1)u'_y \quad ; \quad z_2 + \frac{\beta v}{c^2}(x_2-x_1)u'_z$$

where u'_x, u'_y, u'_z are the velocity components in S'; the components of A'B' are, accordingly,

$$\beta(x_2-x_1)\ (1+vu'_x/c^2) \quad ; \quad y_2-y_1+\frac{\beta v}{c^2}(x_2-x_1)u'_y \quad ; \quad z_2-z_1+\frac{\beta v}{c^2}(x_2-x_1)u'_z$$

It then follows from equations (1.2-2/5) that

$$x'_2-x'_1 = \frac{(x_2-x_1)}{\beta(1-vu_x/c^2)} \tag{1.7-1}$$

$$y'_2-y'_1 = y_2-y_1 + \frac{vu_y}{c^2}\frac{(x_2-x_1)}{(1-vu_x/c^2)} \tag{1.7-2}$$

$$z'_2-z'_1 = z_2-z_1 + \frac{vu_z}{c^2}\frac{(x_2-x_1)}{(1-vu_x/c^2)} \tag{1.7.3}$$

Since t is arbitrary, the result is general.

Consider now the transformation of an elementary parallelepiped having edges parallel to the x,y,z axes in S and lengths $\Delta x, \Delta y, \Delta z$. By applying equations (1.7-1) to 1.7-3) to each edge in turn we find that those edges parallel to the y and z axes remain parallel and do not change in length, while those parallel to the x axis in S remain straight but develop y and z components in S'; further, $\Delta x' = \Delta x\big/\beta(1-vu_x/c^2)$. The volume of the transformed element at any instant in S' is consequently given by

$$\Delta\tau' = \Delta\tau\big/\beta(1-vu_x/c^2) \tag{1.7-4}$$

It is not difficult to show that when a closed surface in S shares the velocity of the sources in which it is immersed, interior sources in S appear as interior sources in S' while exterior sources in S appear as

exterior sources in S'. Hence the instantaneous population density transforms between conjugate points in the inverse ratio of the volume transformation as given above.

The result clearly applies to any type of distribution (ie singlets, doublets or whirls) where individual elements share a common velocity. Its primary application in the present context is to doublet and whirl systems moving in S with velocity $\bar{i}v$, the relevant relationship then being

$$\Delta\tau' = \Delta\tau \Big/ \beta(1-v^2/c^2) = \beta\Delta\tau \qquad (1.7\text{-}4a)$$

EXERCISES

1-19. Prove that in the absence of acceleration a straight line in S transforms to a straight line in S' when viewed at a given instant, and that a point which divides the line in a certain ratio in S maintains this ratio upon transformation.

1-20. Show that interior and exterior points of an elementary parallelepiped which share its velocity in S remain interior and exterior points in S', but that an interior point in S may become an exterior point in S' if it does not share the velocity of the bounding surface.

1-21. On writing the instantaneous population density as D_{inst}, it is seen from equation (1.7-4) that

$$D'_{inst}/D_{inst} = \beta(1-vu_x/c^2)$$

Develop an alternative proof of this in the following way:

With an arbitrary origin of retardation O in S, consider

(1) the instantaneous disposition of sources in a neighbourhood of a point T at the instant t-R/c, where $\bar{R} = \overrightarrow{TO}$.

(2) the individually-retarded disposition of sources in the same region.

Show that the density transformation for individually-retarded sources is given by

$$D'_{ret}/D_{ret} = 1 \Big/ \beta(1-lv/c) \qquad \text{where } l = \bar{i}\cdot\bar{R}/R$$

Bearing in mind the relationship between the population density of an instantaneous configuration and that of its retarded counterpart, as discussed in Sec. 5.17, Pt. 1, deduce that

$$D'_{inst}/D_{inst} = D'_{ret}(1-u_R/c) \Big/ D_{ret}(1-u'_{R'}/c)$$

where u_R and u'_R, are the resolved parts of the source velocities along $\overrightarrow{TO}$ and $\overrightarrow{T'O'}$.

Then employ equation (1.3-6) to complete the demonstration.

Since the value of t and the position of O are arbitrary, the result is general.

1-22. If ρ_1, ρ_2--- are the densities of singlet distributions having the velocities $\bar{u}_1$, $\bar{u}_2$--- in a neighbourhood of a point in S, show that the net singlet density at the conjugate point in S′ is given by

$$\rho' = \beta(\rho - vJ_x/c^2)$$

where $\rho = \rho_1 + \rho_2$--- and J_x is the x component of $\rho_1\bar{u}_1 + \rho_2\bar{u}_2$---. Show further that

$$J'_x = \beta(J_x - \rho v) \quad ; \quad J'_y = J_y \quad ; \quad J'_z = J_z$$

Note that J_x, J_y, J_z, ρ comprise the components of a four-vector.

1-23. Suppose that the filamentary source system of Fig. 1-3 is expanded to form a closed tube which carries a neutral, time-dependent, volume distribution of current. Use the results of the previous exercise to show that at conjugate points of Γ and Γ′

$$J'_x = \beta J_x \quad ; \quad J'_y = J_y \quad ; \quad J'_z = J_z \quad ; \quad \rho' = -\frac{v}{c^2} J'_x \qquad .$$

If, now, the tube is diminished in sectional area while the current density is increased to maintain the current constant at any instant, show that the above relationships give rise, in the limit, to equations (1.5-3) and (1.5-4).

[Note that the current in a contour element is not given by $\bar{J}\cdot d\bar{S}$, where $d\bar{S}$ is the vector area of the section, when the element has a component of velocity parallel to it and the densities of positive and negative sources are unequal. However, this is not the case in Γ or Γ′ in the present instance.]

1.8 The Macroscopic Potentials and their Derivatives

The macroscopic potential functions are required, by definition, to match their microscopic counterparts at points sufficiently removed from source complexes, as discussed in Sec. 5.18, Pt. 1. They have been expressed, in earlier chapters, as integrals involving piecewise continuous density functions and their time derivatives. The analytical prolongation of these expressions within source complexes then serves to define the macroscopic potentials at interior points.

The macroscopic potentials of doublet and whirl distributions which are stationary in S′ may be written down immediately. Following upon the considerations of earlier chapters and the results of Sec. 1.6 we have

(1) doublet distribution:

$$\phi' = \int_{\tau'} \left\{ [\bar{P}'] \cdot \text{grad}' \frac{1}{r'} - \left[\frac{\partial \bar{P}'}{\partial t'}\right] \cdot \frac{\bar{r}'}{cr'^2} \right\} d\tau' \qquad (1.8\text{-}1)$$

$$\bar{A}' = \int_{\tau'} \frac{1}{cr'} \left[\frac{\partial \bar{P}'}{\partial t'}\right] d\tau' \qquad (1.8\text{-}2)$$

where $\bar{P}'$ derives from $\bar{p}'$ as defined by (1.5-2)[4].

(2) whirl distribution:

$$\phi' = \int_{\tau'} \left\{ [\bar{P}'] \cdot \text{grad}' \frac{1}{r'} - \left[\frac{\partial \bar{P}'}{\partial t'}\right] \cdot \frac{\bar{r}'}{cr'^2} \right\} d\tau' \qquad (1.8\text{-}3)$$

$$\bar{A}' = \int_{\tau'} \left\{ [\bar{M}'] \times \text{grad}' \frac{1}{r'} - \left[\frac{\partial \bar{M}'}{\partial t'}\right] \times \frac{\bar{r}'}{cr'^2} + \frac{1}{cr'}\left[\frac{\partial \bar{P}'}{\partial t'}\right] \right\} d\tau' \qquad (1.8\text{-}4)$$

where $\bar{P}'$ and $\bar{M}'$ derive from $\bar{p}'$ and $\bar{m}'$ as defined by (1.5-15) and (1.5-14) respectively[4].

$\bar{r}'$ is the position vector of $d\tau'$ relative to the point of evaluation of ϕ' and $\bar{A}'$.

It is not possible to write down similar relationships for the potentials of the parent distributions in S. Indeed, it is evident from the results of Ex. 1-13. that the microscopic potentials of a uniformly translating doublet are inordinately complicated, and that any attempt to derive a macroscopic form in terms of S parameters would be unproductive.

Nevertheless, it follows from the considerations of Sec. 1.3 that the microscopic potentials at points beyond the source configuration in S are identical with the microscopic - and consequently the macroscopic - potentials at the conjugate points in S′ when combined in the form

$$\phi = \beta(\phi' + \frac{v}{c}A'_x) \qquad (1.8\text{-}5)$$

$$A_x = \beta(A'_x + \frac{v}{c}\phi') \qquad A_y = A'_y \qquad A_z = A'_z \qquad (1.8\text{-}6)$$

---

4. It will not be necessary to make use of the relationships expressed by (1.5-2), (1.5-14) and (1.5-15) as far as individual volume elements are concerned; it is sufficient for our purpose that ϕ' and $\bar{A}'$ assume the above general forms. However, we will subsequently apply $\bar{P}'/\bar{M}'/\bar{P}/\bar{M}$ transformations to conjugate points of evaluation of the field quantities.

We now define the macroscopic potentials at any point O of S as the macroscopic potentials at O' transformed in accordance with (1.8-5/6)[5]. In these circumstances the macroscopic $\bar{E}$ and $\bar{B}$ fields transform in the same way as their microscopic counterparts, ie in accordance with (1.4-11).

It has been shown in Secs.5.19/20,Pt.1, that expressions of the type (1.8-1/4) give rise to $\bar{E}$ and $\bar{B}$ fields which satisfy Maxwell's equations. While polarisation has not been associated previously with a whirl distribution, the additional terms introduced, as a consequence, in (2) above may be supposed to derive from an independent doublet distribution, and, as such, cannot affect the argument.

We have, therefore,

$$\operatorname{div}' \bar{E}' = -4\pi \operatorname{div}' \bar{P}' \tag{1.8-7}$$

$$\operatorname{curl}' \bar{E}' = -\frac{1}{c}\frac{\partial \bar{B}'}{\partial t'} \tag{1.8-8}$$

$$\operatorname{div}' \bar{B}' = 0 \tag{1.8-9}$$

$$\operatorname{curl}' \bar{B}' = \frac{4\pi}{c}\frac{\partial \bar{P}'}{\partial t'} + \frac{1}{c}\frac{\partial \bar{E}'}{\partial t'} + 4\pi \operatorname{curl}' \bar{M}' \tag{1.8-10}$$

We may transform these equations into their equivalents at the conjugate point in S through the agency of the relationships embodied in (1.4-11), together with the inverse of (1.4-1) and (1.4-7), viz

$$\frac{\partial}{\partial x'} = \beta\left(\frac{\partial}{\partial x} + \frac{v}{c^2}\frac{\partial}{\partial t}\right) \tag{1.8-11}$$

$$\frac{\partial}{\partial t'} = \beta\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right) \tag{1.8-12}$$

It only remains to express $\bar{P}'$ and $\bar{M}'$ in terms of $\bar{P}$ and $\bar{M}$.

The relevant population density transformation is given by (1.7-4a). On combining this with (1.5-2), (1.5-14) and (1.5-15) we obtain

(1) doublet distribution

$$P'_x = P_x \qquad P'_y = P_y/\beta \qquad P'_z = P_z/\beta \tag{1.8.13}$$

---

5. It is not difficult to see that this is less a definition than an analytical consequence, if the source structure is sufficiently 'finely-grained'.

(2) whirl distribution

$$M'_x = M_x \qquad M'_y = \beta M_y \qquad M'_z = \beta M_z \tag{1.8-14}$$

$$P'_x = 0 \qquad P'_y = \frac{v}{c} M'_z = \frac{\beta v}{c} M_z \qquad P'_z = -\frac{v}{c} M'_y = -\frac{\beta v}{c} M_y \tag{1.?-15}$$

1.9 Extension of Maxwell's Equations

Since the macroscopic $\bar{E}$ and $\bar{B}$ fields in S are no longer defined directly in terms of macroscopic potentials in S it is not possible to invoke the relationships

$$\bar{E} = -\text{grad}\,\phi - \frac{1}{c}\frac{\partial \bar{A}}{\partial t} \quad ; \quad \bar{B} = \text{curl}\,\bar{A}$$

to prove that

$$\text{curl}\,\bar{E} = -\frac{1}{c}\frac{\partial \bar{B}}{\partial t} \quad \text{and} \quad \text{div}\,\bar{B} = 0$$

These equalities do, in fact, subsist and may be demonstrated as follows.

Since div′ $\bar{B}' = 0$ it follows from (1.4-11) and (1.8-11) that

$$\frac{\partial B'_x}{\partial x'} + \frac{\partial B'_y}{\partial y'} + \frac{\partial B'_z}{\partial z'} = \beta\left(\frac{\partial B_x}{\partial x} + \frac{v}{c^2}\frac{\partial B_x}{\partial t}\right) + \frac{\partial}{\partial y}\beta(B_y + \frac{v}{c}E_z) + \frac{\partial}{\partial z}\beta(B_z - \frac{v}{c}Ey) = 0$$

$$\text{whence} \qquad \text{div}\,\bar{B} + \frac{v}{c}(\text{curl}\,\bar{E})_x + \frac{v}{c^2}\frac{\partial B_x}{\partial t} = 0 \tag{1.9-1}$$

$$\text{Also} \qquad (\text{curl}'\bar{E}')_x = -\frac{1}{c}\frac{\partial B'_x}{\partial t'}$$

$$\text{or} \quad \frac{\partial E'_z}{\partial y'} - \frac{\partial E'_y}{\partial z'} = \frac{\partial}{\partial y}\beta\left(E_z + \frac{v}{c}B_y\right) - \frac{\partial}{\partial z}\beta\left(E_y - \frac{v}{c}B_z\right) = -\frac{1}{c}\beta\left(\frac{\partial B_x}{\partial t} + v\frac{\partial B_x}{\partial x}\right)$$

$$\text{whence} \qquad (\text{curl}\,\bar{E})_x + \frac{1}{c}\frac{\partial B_x}{\partial t} + \frac{v}{c}\text{div}\,\bar{B} = 0 \tag{1.9-2}$$

On substituting for $\frac{v}{c}$ div $\bar{B}$ from (1.9-1) we get

$$\text{div } \bar{B} = 0 \tag{1.9-3}$$

and

$$(\text{curl } \bar{E})_x = -\frac{1}{c}\frac{\partial B_x}{\partial t}$$

It may be shown in a similar manner that $(\text{curl } \bar{E})_y = -\frac{1}{c}\frac{\partial B_y}{\partial t}$ and $(\text{curl } \bar{E})_z = -\frac{1}{c}\frac{\partial B_z}{\partial t}$, hence

$$\text{curl } \bar{E} = -\frac{1}{c}\frac{\partial \bar{B}}{\partial t} \tag{1.9-4}$$

We now turn to the transformation of the remaining equations. The general form will be developed first; substitution for $\bar{P}'$ and $\bar{M}'$, as set out in (1.8-13) to (1.8-15), will be undertaken subsequently.

We have

$$\text{div}' \bar{E}' = -4\pi \text{ div}'\bar{P}' \tag{1.9-5}$$

or

$$\frac{\partial E'_x}{\partial x'} + \frac{\partial E'_y}{\partial y'} + \frac{\partial E'_z}{\partial z'} = -4\pi\left(\frac{\partial P'_x}{\partial x'} + \frac{\partial P'_y}{\partial y'} + \frac{\partial P'_z}{\partial z'}\right)$$

On transforming each side we obtain

$$\beta \text{ div } \bar{E} + \frac{\beta v}{c^2}\frac{\partial E_x}{\partial t} - \frac{\beta v}{c}(\text{curl } \bar{B})_x = -4\pi\beta\left(\frac{\partial P'_x}{\partial x} + \frac{v}{c^2}\frac{\partial P'_x}{\partial t}\right) - 4\pi\left(\frac{\partial P'_y}{\partial y} + \frac{\partial P'_z}{\partial z}\right)$$

or

$$\text{div } \bar{E} = \frac{v}{c}(\text{curl } \bar{B})_x - \frac{v}{c^2}\frac{\partial E_x}{\partial t} - 4\pi\left(\frac{\partial P'_x}{\partial x} + \frac{v}{c^2}\frac{\partial P'_x}{\partial t}\right) - \frac{4\pi}{\beta}\left(\frac{\partial P'_y}{\partial y} + \frac{\partial P'_z}{\partial z}\right) \tag{1.9-6}$$

We have also

$$(\text{curl}' \bar{B}')_x = \frac{4\pi}{c}\frac{\partial P'_x}{\partial t'} + \frac{1}{c}\frac{\partial E'_x}{\partial t'} + 4\pi(\text{curl}' \bar{M}')_x \tag{1.9-7}$$

$$\text{or} \quad \frac{\partial B'_z}{\partial y'} - \frac{\partial B'_y}{\partial z'} = \frac{4\pi}{c}\frac{\partial P'_x}{\partial t'} + \frac{1}{c}\frac{\partial E'_x}{\partial t'} + 4\pi\left(\frac{\partial M'_z}{\partial y'} - \frac{\partial M'_y}{\partial z'}\right)$$

whence we find that

$$\beta(\text{curl } \bar{B})_x - \frac{\beta v}{c}\,\text{div } \bar{E} + \frac{\beta v}{c}\frac{\partial E_x}{\partial x} = \frac{4\pi\beta}{c}\left(\frac{\partial P'_x}{\partial t} + v\frac{\partial P'_x}{\partial x}\right) + \frac{\beta}{c}\left(\frac{\partial E_x}{\partial t} + v\frac{\partial E_x}{\partial x}\right) + 4\pi\left(\frac{\partial M'_z}{\partial y} - \frac{\partial M'_y}{\partial z}\right)$$

or

$$(\text{curl } \bar{B})_x = \frac{v}{c}\,\text{div } \bar{E} - \frac{v}{c}\frac{\partial E_x}{\partial x} + \frac{4\pi}{c}\left(\frac{\partial P'_x}{\partial t} + v\frac{\partial P'_x}{\partial x}\right) + \frac{1}{c}\left(\frac{\partial E_x}{\partial t} + v\frac{\partial E_x}{\partial x}\right) + \frac{4\pi}{\beta}\left(\frac{\partial M'_z}{\partial y} - \frac{\partial M'_y}{\partial z}\right) \tag{1.9-8}$$

On substituting for $(\text{curl } \bar{B})_x$ in (1.9-6) we obtain

$$\text{div } \bar{E} = -4\pi\left\{\frac{\partial P'_x}{\partial x} + \frac{\partial}{\partial y}\beta\left(P'_y - \frac{v}{c}M'_z\right) + \frac{\partial}{\partial z}\beta\left(P'_z + \frac{v}{c}M'_y\right)\right\} \tag{1.9-9}$$

and

$$(\text{curl } \bar{B})_x = \frac{4\pi}{c}\frac{\partial P'_x}{\partial t} + \frac{1}{c}\frac{\partial E_x}{\partial t} + 4\pi\left\{\frac{\partial}{\partial y}\beta\left(M'_z - \frac{v}{c}P'_y\right) - \frac{\partial}{\partial z}\beta\left(M'_y + \frac{v}{c}P'_z\right)\right\} \tag{1.9.10}$$

In similar manner we find that

$$(\text{curl } \bar{B})_y = \frac{4\pi}{c}\frac{\partial}{\partial t}\beta\left(P'_y - \frac{v}{c}M'_z\right) + \frac{1}{c}\frac{\partial E_y}{\partial t} + 4\pi\left\{\frac{\partial M'_x}{\partial z} - \frac{\partial}{\partial x}\beta\left(M'_z - \frac{v}{c}P'_y\right)\right\} \tag{1.9-11}$$

$$(\text{curl } \bar{B})_z = \frac{4\pi}{c}\frac{\partial}{\partial t}\beta\left(P'_z + \frac{v}{c}M'_y\right) + \frac{1}{c}\frac{\partial E_z}{\partial t} + 4\pi\left\{\frac{\partial}{\partial x}\beta\left(M'_y + \frac{v}{c}P'_z\right) - \frac{\partial M'_x}{\partial y}\right\} \tag{1.9-12}$$

On substituting for $\bar{P}'$ and $\bar{M}'$ in terms of $\bar{P}$ and $\bar{M}$ we obtain

$$\text{div } \bar{E} = -4\pi\left(\frac{\partial P_x}{\partial x} + \frac{\partial P_y}{\partial y} + \frac{\partial P_z}{\partial z}\right) = -4\pi\,\text{div } \bar{P} \tag{1.9-13}$$

$$(\mathrm{curl}\ \bar{B})_x = \frac{4\pi}{c}\frac{\partial P_x}{\partial t} + \frac{1}{c}\frac{\partial E_x}{\partial t} + 4\pi\left\{\frac{\partial}{\partial y}\left(M_z - \frac{v}{c}P_y\right) - \frac{\partial}{\partial z}\left(M_y + \frac{v}{c}P_z\right)\right\} \tag{1.9-14}$$

$$= \frac{4\pi}{c}\frac{\partial P_x}{\partial t} + \frac{1}{c}\frac{\partial E_x}{\partial t} + 4\pi(\mathrm{curl}\ \bar{M})_x - \frac{4\pi v}{c}\left(\frac{\partial P_y}{\partial y} + \frac{\partial P_z}{\partial z}\right) \tag{1.9-14a}$$

$$(\mathrm{curl}\ \bar{B})_y = \frac{4\pi}{c}\frac{\partial P_y}{\partial t} + \frac{1}{c}\frac{\partial E_y}{\partial t} + 4\pi\left\{\frac{\partial M_x}{\partial z} - \frac{\partial}{\partial x}\left(M_z - \frac{v}{c}P_y\right)\right\} \tag{1.9-15}$$

$$= \frac{4\pi}{c}\frac{\partial P_y}{\partial t} + \frac{1}{c}\frac{\partial E_y}{\partial t} + 4\pi\ (\mathrm{curl}\ \bar{M})_y + \frac{4\pi v}{c}\frac{\partial P_y}{\partial x} \tag{1.9-15a}$$

$$(\mathrm{curl}\ \bar{B})_z = \frac{4\pi}{c}\frac{\partial P_z}{\partial t} + \frac{1}{c}\frac{\partial E_z}{\partial t} + 4\pi\left\{\frac{\partial}{\partial x}\left(M_y + \frac{v}{c}P_z\right) - \frac{\partial M_x}{\partial y}\right\} \tag{1.9-16}$$

$$= \frac{4\pi}{c}\frac{\partial P_z}{\partial t} + \frac{1}{c}\frac{\partial E_z}{\partial t} + 4\pi\ (\mathrm{curl}\ \bar{M})_z + \frac{4\pi v}{c}\frac{\partial P_z}{\partial x} \tag{1.9-16a}$$

Equations (1.9-14a), (1.9-15a) and (1.9-16a) may be subsumed under the single equation

$$\mathrm{curl}\ \bar{B} = \frac{4\pi}{c}\frac{\partial \bar{P}}{\partial t} + \frac{1}{c}\frac{\partial \bar{E}}{\partial t} + 4\pi\ \mathrm{curl}\ \bar{M} + \frac{4\pi}{c}\ \mathrm{curl}\ (\bar{P} \times \bar{v}) \tag{1.9-17}$$

If we add to the above relationships the terms which derive from the presence of singlet distributions, as developed in Part 1, we obtain the general equations

$$\mathrm{div}\ \bar{E} = 4\pi\ (\rho - \mathrm{div}\ \bar{P}) \tag{1.9-18}$$

$$\text{or} \quad \mathrm{div}\ \bar{D} = 4\pi\ \rho \tag{1.9-18a}$$

$$\mathrm{curl}\ \bar{E} = -\frac{1}{c}\frac{\partial \bar{B}}{\partial t} \tag{1.9-4}$$

$$\mathrm{div}\ \bar{B} = 0 \tag{1.9-3}$$

$$\text{curl } \bar{B} = \frac{4\pi}{c}\bar{J} + \frac{4\pi}{c}\frac{\partial \bar{P}}{\partial t} + \frac{1}{c}\frac{\partial \bar{E}}{\partial t} + 4\pi \text{ curl } \bar{M} + \frac{4\pi}{c}\text{ curl } (\bar{P} \times \bar{v}) \qquad (1.9\text{-}19)$$

or

$$\text{curl } \bar{H} = \frac{4\pi}{c}\bar{J} + \frac{1}{c}\frac{\partial \bar{D}}{\partial t} + \frac{4\pi}{c}\text{ curl } (\bar{P} \times \bar{v}) \qquad (1.9\text{-}19a)$$

It should be noted that $\bar{J}$ involves absolute velocities in S, and consequently includes components which, in the present context, could be considered to be convection currents.

We may replace $\bar{v}$ in equation (1.9-19a) by the more general velocity $\bar{u}$ since the axes of coordinates may always be so orientated as to bring the x axis into line with $\bar{u}$. However, the application is not restricted to uniform translation in a single direction at a given time. For a given point of evaluation it is possible to consider different configurations separately and combine the results by superposition since the equations are linear and div and curl are invariant with respect to choice of axes. In this case $\bar{u}$ is, of course, the value of velocity associated with the local doublet distribution.

No restriction has been placed upon the value of u other than $|u|<c$, but acceleration is not permitted because this would lead to as-yet-unknown modifications of the expressions for the macroscopic potentials in S'.

It will be seen that when a configuration of translating doublets replaces one of stationary whirls, the term 4π curl $(\bar{P}\times\bar{v}/c)$ replaces 4π curl $\bar{M}$ in the expression for curl $\bar{B}$. This may lead one to suppose that a doublet of moment $\bar{p}$ has an associated whirl moment $\bar{p}\times\bar{v}/c$. If this were the case we would indeed obtain the macroscopic relationship, but the reverse does not follow, and the equivalence clearly fails at microscopic level. Thus the vector potential of the doublet is directed along the line of its velocity while that of the whirl, when viewed in its own plane, is normal to the radius vector drawn to the point of evaluation. Furthermore, the $\bar{B}$ fields are unequal; the radial field of the doublet is only half that of the whirl, for small values of v/c, when viewed along the normal to $\bar{p}$ and $\bar{v}$. (Ex. 1-24)

1.10 Extension of Boundary Conditions

The boundary conditions for stationary surfaces of discontinuity in S' are obtained by priming equations (5.21-16) to (5.21-19a), Pt. 1. We have, inter alia,

$$\Delta\left[\hat{n}'\cdot(\bar{E}' + 4\pi\,\bar{P}')\right] = 4\pi\sigma' \qquad (1.10\text{-}1)$$

$$\Delta\left[\hat{n}'\times\bar{E}'\right] = 0 \qquad (1.10\text{-}2)$$

$$\Delta\left[\hat{n}'\cdot\bar{B}'\right] = 0 \qquad (1.10\text{-}3)$$

$$\Delta\left[\hat{n}'\times(\bar{B}' - 4\pi\bar{M}')\right] = \frac{4\pi}{c}\bar{K}' \qquad (1.10\text{-}4)$$

If $d\bar{S}$ and $d\bar{S}'$ are conjugate surface elements, and $d\bar{S}$ translates with velocity $\bar{i}v$ in S

$$dS'_x = dS_x \qquad dS'_y = \beta dS_y \qquad dS'_z = \beta dS_z$$

or

$$\hat{n}'_x dS' = \hat{n}_x dS \qquad \hat{n}'_y dS' = \beta\hat{n}_y dS \qquad \hat{n}'_z dS' = \beta\hat{n}_z dS$$

where $\hat{\bar{n}}$ and $\hat{\bar{n}}'$ are the unit normals.
Then if we write $dS/dS'=\alpha$ we have

$$\hat{n}'_x = \alpha\hat{n}_x \qquad \hat{n}'_y = \alpha\beta\hat{n}_y \qquad \hat{n}'_z = \alpha\beta\hat{n}_z \tag{1.10-5}$$

Suppose that the net densities of stationary surface sources on dS and dS' are designated σ_s and σ'_s while those of the moving sources comprising the surface currents are designated σ_m and σ'_m. Then $\bar{K}=\sigma_m\bar{w}$ and $\bar{K}'=\sigma'_m\bar{w}'$ where $\bar{w}$ and $\bar{w}'$ are the velocities of the sources relative to dS and dS'.

Let the line $\overrightarrow{AB}$ be embedded in dS and directed along the line of surface current $\bar{K}$. Suppose that the coordinates of A and B at the time t are x_1+vt, y_1, z_1 and x_2+vt, y_2, z_2. Then a source which reaches A at the time t transforms to an equal source at $\beta x_1, y_1, z_1$ at the time $\beta\left\{t-v(x_1+vt)/c^2\right\}$ = say, t', while a source reaching B at time t transforms to $\beta x_2, y_2, z_2$ at the time $\beta\left\{t-v(x_2+vt)/c^2\right\} = t'-\beta v(x_2-x_1)/c^2$. At the time t' the latter source will have moved to B'' where $\overrightarrow{B'B''} = \beta v(x_2-x_1)\bar{w}'/c^2$. Hence the spacing of moving sources along the line A'B' at any instant is increased over that of stationary sources by the ratio

$$\frac{A'B''}{A'B'} = \frac{(A'B'')_x}{(A'B')_x} = \left\{\beta(x_2-x_1)+\frac{\beta v}{c^2}(x_2-x_1)w'_x\right\}\Big/\beta(x_2-x_1) = 1+vw'_x/c^2$$

It then follows that $\sigma'_s=\alpha\sigma_s$ and $\sigma'_m=\alpha\sigma_m/(1+vw'_x/c^2)$

so that if σ and σ' are the overall source densities upon dS and dS'

$$\sigma=\sigma_s+\sigma_m \qquad \sigma'=\alpha\left\{\sigma_s+\sigma_m/(1+vw'_x/c^2)\right\} \tag{1.10-6}$$

$$\bar{K}=\sigma_m\bar{w} \qquad \bar{K}'=\alpha\sigma_m\bar{w}'/(1+vw'_x/c^2) \tag{1.10-7}$$

We are now in a position to transform equations (1.10-1) to (1.10-4).

From (1.10-2), $\Delta(\hat{\bar{n}}' \times \bar{E}')_x = 0$

Substitution from (1.4-11) and (1.10-5) then yields

$$\Delta\left[\alpha\left\{\beta\hat{n}_y\beta(E_z + \frac{v}{c}B_y) - \beta\hat{n}_z\beta(E_y - \frac{v}{c}B_z)\right\}\right] = 0$$

or

$$\Delta\left[\alpha\left\{\beta^2(\hat{\bar{n}} \times \bar{E})_x + \beta^2\frac{v}{c}(\hat{n}_yB_y + \hat{n}_zB_z)\right\}\right] = 0$$

$$\text{ie}\qquad \Delta\left[(\hat{\bar{n}} \times \bar{E})_x + \frac{v}{c}(\hat{n}_yB_y + \hat{n}_zB_z)\right] = 0 \qquad (1.10\text{-}8)$$

Similarly, expansions of $\Delta(\hat{\bar{n}}' \times \bar{E}')_y = 0$ and $\Delta(\hat{\bar{n}}' \times \bar{E}')_z = 0$ yield

$$\Delta\left[(\hat{\bar{n}} \times \bar{E})_y - \frac{v}{c}\hat{n}_xB_y\right] = 0 \qquad (1.10\text{-}9)$$

and

$$\Delta\left[(\hat{\bar{n}} \times \bar{E})_z - \frac{v}{c}\hat{n}_xB_z\right] = 0 \qquad (1.10\text{-}10)$$

It is easily shown that equations (1.10-8) to (1.10-10) can be subsumed under the single equality

$$\Delta\left[\hat{\bar{n}} \times \left(\bar{E} + (\frac{\bar{v}}{c} \times \bar{B})\right)\right] = 0 \qquad (1.10\text{-}11)$$

From the expansion of (1.10-3) we find that

$$\Delta\left[\hat{\bar{n}}\cdot\bar{B} + \frac{\beta^2v^2}{c^2}(\hat{n}_yB_y + \hat{n}_zB_z) + \beta^2\frac{v}{c}(\hat{\bar{n}} \times \bar{E})_x\right] = 0 \qquad (1.10\text{-}12)$$

On multiplying (1.10-8) by $\beta^2\frac{v}{c}$ and combining with (1.10-12) we get

$$\Delta\,(\hat{\bar{n}}\cdot\bar{B}) = 0 \qquad (1.10\text{-}13)$$

Expansion of the x component of (1.10-4) yields

$$\Delta\left[\beta^2\left(\hat{n}\times(\bar{B}-4\pi\bar{M})\right)_x-\beta^2\frac{v}{c}\,(\hat{n}_yE_y+\hat{n}_zE_z)\right]=\frac{4\pi}{c}\,\sigma_m w'_x/(1+vw'_x/c^2)$$

$$\text{But } w'_x=\frac{(w_x+v)-v}{1-v(w_x+v)/c^2}\qquad\text{whence}\qquad\frac{w'_x}{(1+vw'_x/c^2)}=\beta^2 w_x \tag{1.10-14}$$

so that

$$\Delta\left[\left(\hat{n}\times(\bar{B}-4\pi\bar{M})\right)_x-\frac{v}{c}\,(\hat{n}_yE_y+\hat{n}_zE_z)\right]=\frac{4\pi}{c}\,K_x \tag{1.10-15}$$

From the expansion of the y component of (1.10-4) we get

$$\Delta\left[\beta\left(\hat{n}\times(\bar{B}-4\pi\bar{M})\right)_y+\beta\frac{v}{c}\,\hat{n}_xE_y\right]=\frac{4\pi}{c}\,\sigma_m w'_y/(1+vw'_x/c^2)$$

But $w'_y\,/(1+\frac{v}{c^2}\,w'_x)$ reduces to βw_y, hence

$$\Delta\left[\left(\hat{n}\times(\bar{B}-4\pi\bar{M})\right)_y+\frac{v}{c}\,\hat{n}_xE_y\right]=\frac{4\pi}{c}\,K_y \tag{1.10-16}$$

Similarly

$$\Delta\left[\left(\hat{n}\times(\bar{B}-4\pi\bar{M})\right)_z+\frac{v}{c}\,\hat{n}_xE_z\right]=\frac{4\pi}{c}\,K_z \tag{1.10-17}$$

Equations (1.10-15) to (1.10-17) can be subsumed under

$$\Delta\left[\hat{n}\times\left(\bar{B}-4\pi\bar{M}-(\frac{\bar{v}}{c}\times\bar{E})\right)\right]=\frac{4\pi}{c}\,\bar{K} \tag{1.10-18}$$

Finally, the expansion of (1.10-1) yields

$$\Delta\left[\hat{n}\cdot(\bar{E}+4\pi\bar{P})+\frac{\beta^2v^2}{c^2}\,(\hat{n}_yE_y+\hat{n}_zE_z)-\frac{\beta^2v}{c}\left(\hat{n}\times(\bar{B}-4\pi\bar{M})\right)_x\right]=4\pi\left\{\sigma_s+\sigma_m/(1+\frac{v}{c^2}\,w'_x)\right\} \tag{1.10-19}$$

By multiplying (1.10-15) by $\beta^2 \frac{v}{c}$ and combining with (1.10-19) we get

$$\Delta\left[\hat{\bar{n}}\cdot(\bar{E} + 4\pi\bar{P})\right] = 4\pi \left\{\sigma_s + \frac{\sigma_m}{(1 + \frac{v}{c^2} W'_x)} + \beta^2 \frac{v}{c^2} K_x\right\}$$

The right hand side of this equation reduces to $4\pi(\sigma_s + \sigma_m)$ hence

$$\Delta\left[\hat{\bar{n}}\cdot(\bar{E}+4\pi\bar{P})\right] = 4\pi\sigma \qquad (1.10\text{-}20)$$

In summary,

$$\Delta\left[\hat{\bar{n}}\cdot\bar{E}\right] = 4\pi\ (\sigma - \Delta\ \hat{\bar{n}}\cdot\bar{P}\ ) \qquad (1.10\text{-}20)$$

or

$$\Delta\left[\hat{\bar{n}}\cdot\bar{D}\right] = 4\pi\sigma \qquad (1.10\text{-}20a)$$

$$\Delta\left[\hat{\bar{n}}\cdot\bar{B}\right] = 0 \qquad (1.10\text{-}13)$$

$$\Delta\left[\hat{\bar{n}} \times \left(\bar{E} + \frac{1}{c}\ (\bar{u} \times \bar{B})\right)\right] = 0 \qquad (1.10\text{-}11)$$

$$\Delta\ \left[\hat{\bar{n}} \times \left(\bar{B} - 4\pi\bar{M} - \frac{1}{c}\ (\bar{u} \times \bar{E})\right)\right] = \frac{4\pi}{c}\ \bar{K} \qquad (1.10\text{-}18)$$

or

$$\Delta\left[\hat{\bar{n}} \times \left(\bar{H} - \frac{1}{c}\ (\bar{u} \times \bar{E})\right)\right] = \frac{4\pi}{c}\ \bar{K} \qquad (1.10\text{-}18a)$$

where σ, $\bar{K}$, $\bar{P}$ and $\bar{M}$ are the density values obtaining at or on the surface of discontinuity under consideration and $\bar{u}$ is its velocity.

EXERCISES

1-24. A time-invariant doublet in uniform translation with velocity $\bar{v}$ occupies the position Q at a certain time. Its $\bar{B}$ field is to be evaluated at O at this time. If $\overrightarrow{QO} = \bar{r}$ and θ is the angle between $\bar{v}$ and $\bar{r}$, utilise equation (5.11-33), Pt.1, to show that

$$\bar{B} = \left(1 - \frac{v^2}{c^2}\right)\left\{\frac{\bar{p} \times \bar{v}}{cr^3\gamma^3} + \frac{3(\bar{v} \times \bar{r})\ \bar{p}\cdot\bar{r}}{cr^5\gamma^5} + \frac{3(\bar{v} \times \bar{r})\ (\bar{p} \times \bar{v})\cdot(\bar{v} \times \bar{r})}{c^3 r^5 \gamma^5}\right\}$$

$$\text{where } \gamma = \left(1 - \frac{v^2}{c^2} \sin^2 \theta\right)^{1/2}$$

A stationary, time-invariant whirl of moment $\frac{1}{c}(\bar{p} \times \bar{v})$ occupies the position Q. Show, by substitution in equation (5.16-9) Pt. 1, that its $\bar{B}$ field at O is given by

$$\bar{B} = -\frac{\bar{p} \times \bar{v}}{cr^3} + \frac{3\ (\bar{p} \times \bar{v})\cdot\bar{r}\ \bar{r}}{cr^5}$$

Hence show that at points upon the normal to $\bar{p}$ and $\bar{v}$ the radial field of the doublet is half that of the whirl, for $\frac{v}{c} \ll 1$.

1-25. Utilise the relationship div′ $\bar{A}' = -\frac{1}{c}\frac{\partial\phi'}{\partial t'}$ to show that div $\bar{A} = -\frac{1}{c}\frac{\partial\phi}{\partial t}$.

1-26. We have been concerned in Sec. 1.5 with the transformation of doublet and whirl moment densities from systems moving with velocity $\bar{i}v$ in S to stationary systems in S′. Now suppose that the systems are stationary in S and move with velocity $-\bar{i}v$ in S′.

Show that the associated transformations are

$$P'_x = P_x \qquad P'_y = \beta\left(P_y + \frac{v}{c} M_z\right) \qquad P'_z = \beta\left(P_z - \frac{v}{c} M_y\right)$$

$$M'_x = M_x \qquad M'_y = M_y/\beta \qquad M'_z = M_z/\beta$$

1-27. In order to provide a link between the purely algebraical treatment of the foregoing chapter and the applied mathematics of special relativity, it is necessary to identify our point sources with electrical charges and to utilise the Lorentz force law. This states that the force which acts upon a point charge in the presence of other charges is given by

$$\bar{F} = a\left(\bar{E} + \frac{1}{c}(\bar{u} \times \bar{B})\right)$$

where a is the strength of the charge and $\bar{u}$ its velocity, and where $\bar{E}$ and $\bar{B}$ derive from all charges other than that under consideration.

Then the force acting upon the conjugate charge in the S′ configuration (treated as a physical system) is given by

$$\bar{F}' = a\left(\bar{E}' + \frac{1}{c}(\bar{u}' \times \bar{B}')\right)$$

Show that the x component of $\bar{F}'$ may be written as

$$a\left\{E_x + \frac{1}{c}(\bar{u} \times \bar{B})_x - \frac{v}{c^2}\,\bar{u}\cdot\left(\bar{E} + \frac{1}{c}(\bar{u} \times \bar{B})\right)\right\} \Big/ (1 - vu_x/c^2)$$

or
$$F'_x = F_x - \frac{v/c^2}{(1 - vu_x/c^2)}(u_y F_y + u_z F_z)$$

Show also that

$$F'_y = \frac{F_y}{\beta(1 - vu_x/c^2)} \quad ; \quad F'_z = \frac{F_z}{\beta(1 - vu_x/c^2)}$$

1-28. By reference to the results of Ex. 1-6. show that the component forces derived above are precisely those required to produce the rate of change of momentum of the conjugate source as demanded by the purely kinematical transformation (assuming that the mass/velocity ratio may be taken as an experimental fact).

Since this holds for each charge in turn, it follows that the application of the Lorentz transformation to an existing time-dependent configuration of point charges, which is subject only to internal forces of interaction, gives rise, for any value of $|v|<c$, to a further physically-realizable time-dependent configuration.

1-29. The conclusion reached in the previous exercise, when combined with the results of Ex. 1-26, suggests that if $\bar{P}^\circ$ and $\bar{M}^\circ$ are the moment densities of doublets and whirls having stationary centres in S, the values assumed by these densities when the systems move as a whole in S with velocity $\bar{i}v$ will be given by

$$P_x = P^\circ_x \qquad P_y = \beta(P^\circ_y - \frac{v}{c}M^\circ_z) \qquad P_z = \beta(P^\circ_z + \frac{v}{c}M^\circ_y)$$
$$(1)$$
$$M_x = M^\circ_x \qquad M_y = M^\circ_y/\beta \qquad M_z = M^\circ_z/\beta$$

Assuming this to be so, show by substitution in (1.9-13) to (1.9-16) that the equations for div $\bar{E}$ and curl $\bar{B}$ assume their Maxwellian form for systems at rest, provided that we write

$$P_x = P_x^o \qquad P_y = \beta(P_y^o - \frac{v}{c} M_z^o) \qquad P_z = \beta(P_z^o + \frac{v}{c} M_y^o) \tag{2}$$

$$M_x = M_x^o \qquad M_y = \beta(M_y^o + \frac{v}{c} P_z^o) \qquad M_z = \beta(M_z^o - \frac{v}{c} P_y^o)$$

Students of the special theory of relativity will be familiar with these relationships.

1-30. In texts on relativity the left hand side of equation (1.10-18) is replaced by

$$\Delta \left[\hat{n} \times \left((\bar{B}-4\pi\bar{M}) - \frac{\bar{v}}{c} \times (\bar{E}+4\pi\bar{P}) \right) \right]$$

where $\bar{P}$ and $\bar{M}$ are defined by (2) in the previous exercise. Show by substitution that this is identical with (1.10-18) when the value of $\bar{M}$ as defined by (1) is adopted and $\bar{v}$ is replaced by $\bar{u}$.

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17 SUMMARY OR ABSTRACT
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Part 2 of "Field Analysis and Potential Theory" complements the treatment of Maxwell's equations for systems of doublets and whirls at rest, as presented in Part 1. Considerations are extended to time-dependent configurations of doublets and whirls in uniform translation, and the appropriately-modified forms of Maxwell's equations and the boundary conditions are developed. As in Part 1 the Lienard-Wiechert potentials are taken to be fundamental.

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